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# SURREALIST LANDSCAPE WITH FIGURES (A SURVEY OF RECENT RESULTS IN SET THEORY)

by

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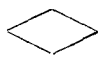
This is the text of the hitherto unpublished article written at Stanford University and circulated in typescript in 1968 with the title page given below. "Recent" being no longer a true description of its contents, I publish it now under a timeless title by which it has been cited by other writers. In doing so, I have corrected the typographical errors in the original text but not the mathematical ones, to which attention is drawn instead in the notes on subsequent developments which will be found together with a supplementary bibliography at the end. The sign + in the text marks points on which those notes comment.

I record here my gratitude to DANA SCOTT, then Professor of Logic at Stanford, who enabled me to spend eight stimulating months in California and whose brainchild the survey was.



## Section 0. Introduction

The present work unfolds the development of Zermelo—Fraenkel set theory in the years following the invention of forcing. Theorems, definitions and unsolved problems are indicated by the appropriate letter. In order to avoid decimal points, statements are numbered in thousands; thus 1309 indicates the ninth theorem of § 3 of Section 1. Proofs are few, and I should state that the observations called theorems vary from trivial remarks to the deepest results in the subject. I think that only the detailed proofs of the theorems would convey adequately their relative importance, and accordingly I have not tried.

The names of those who have worked on the same problem are separated by a comma if they worked together, and by a semicolon if independently. (See, for example, T 1308, which was proved by two independent groups.)

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There are three styles of reference: those of the form [B10] are found at the end, those like [H44] are to be found in the revised Bibliography of the Prague Seminar compiled by HÁJEK, and an occurrence of [*] shortly after people's names refers to the account of their lectures at the Summer Institute expected to appear in the Proceedings.

I assume familiarity with the basic properties of forcing and Boolean-valued models, for which I use the notation of SCOTT and SOLOVAY [*]: general lemmata on forcing are given in Boolean terminology but individual constructions are described in terms I think most vivid. Independence results are stated usually in the form "Con($A$) implies Con($B$)", meaning that the consistency of the system $B$ can be deduced from that of $A$ in elementary number theory. Model theoretic forms are preferred when they give finer information. The systems in which other results are proved are either stated, e.g., "ZF + AC $\vdash$ . . ." or are to be inferred from the immediate context. I write "c.s.m." for "countable standard model"; and I use "a real" always to mean "a subset of $\omega$." AC is the axoim of choice in the form "Every set has a well-ordering."

Each section is divided into paragraphs. The first section starts from GÖDEL's L: in § 1, GCH is true; in § 2, GCH is false but AC is preserved; in § 3, AC too succumbs. The second section is devoted to large cardinals: no mention is made of those of REINHARDT [*], but it is hoped that [B15] will appear soon. Proofs of many of the theorems of Section 2, in particular of TT 2012, 2017, 2018 and 2033, are contained in the lecture notes [B16] of JENSEN on large cardinals which are expected to become accessible this year. Section 3 is on the real line, and includes some result on real-valued measurable cardinals and on the axiom of determinacy. Sections 4 and 5 are brief summaries of work on problems arising in model theory, and on systems other than ZF.

## Section 1. The Axiom of Choice and Continuum Hypothesis

### § 0. *Before Forcing*

T 1000 (GÖDEL). Con(ZF) implies Con (ZF + $V = L$).

T 1001 (GÖDEL). ZF $\vdash V = L \rightarrow$ GCH.

T 1002 (LINDENBAUM–TARSKI). ZF $\vdash$ GCH $\rightarrow$ AC.

\+ § 1. *After Forcing*: *Non-constructibility*

T 1100 (COHEN). Con(ZF) implies

Con(ZF + GCH + there is a non-constructible subset of $\omega$).

Let $2 = \{0, 1\}$ be given the discrete topology and the measure defined by $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$; let $2^\omega$ be given the product topology and measure.

D 1101. $\mathbf{B}_1$ = algebra of Borel subsets of $2^\omega$ modulo sets of first category.

D 1102. $\mathbf{B}_2$ = algebra of Borel subsets of $2^\omega$ modulo sets of measure zero.

Define an element of $V^{(\mathbf{B}_1)}$ ($V^{(\mathbf{B}_2)}$) by

$$[\check{n}\varepsilon\dot{a}]^{\mathbf{B}_1} = \{f \in 2^\omega \mid f(n) = 1\}/\text{sets of first category}$$

$$([\check{n}\varepsilon\dot{a}]^{\mathbf{B}_2} = \{f \in 2^\omega \mid f(n) = 1\}/\text{sets of measure 0}).$$

Let $M$ be a c.s.m. of ZF + AC. Then

D 1103. A real $a_1$ is *generic* over $M$ iff there is a $M$-complete ultrahomomorphism $F_1$ of $\mathbf{B}_1^M$ with

$$a_1 = \{n \mid F_1([\check{n}\varepsilon\dot{a}]^{\mathbf{B}_1^M}) = 1\};$$

+ D 1104. A real $a_2$ is *random* over $M$ iff there is an $M$-complete ultrahomomorphism $F_2$ of $\mathbf{B}_2^M$ with

$$a_2 = \{n \mid F_2([\check{n}\varepsilon\dot{a}]^{\mathbf{B}_2^M}) = 1\};$$

we write $M^{(\mathbf{B}_M^1)}/F_1 = M[a_1]$, etc. For details, see the paper of SCOTT and SOLOVAY [*].

Cohen proved T 1100 by showing that $M[a_1] \vdash \dot{a}_1 \varepsilon L$, where $M$ is a v.s.m. of ZF + $V = L$, and $a_1$ is generic over $M$. The concept of "random real" is due to Solovay, who also showed that D 1103 is equivalent to the notion of generic real implicit in Cohen's proof. We shall see later that $M[a_1]$ and $M[a_2]$, where $a_2$ is random over $M$, differ in many details; for the present, let us record Solovay's observation that $M[a_1]$ contains no reals random and $M[a_2]$ no reals generic over $M$.

We shall write simply "random" or "generic", omitting "over $M$", during discussions in which the ground model $M$ is fixed.

Now that we know that non-constructible reals are possible, we may consider "degrees of constructibility", or *L-degrees*, for real numbers:

D 1105. $a \leq_L b$ iff $a$ is constructible in $b$; that is $a \in L[b]$.

Sacks invented a method of forcing which yielded a model having precisely two $L$-degrees. Thus

+ T 1106 (SACKS). Con(ZF) implies

$$\text{Con}(\text{ZF} + \text{GCH} + \text{there is a non-constructible real} +$$

$$+ \bigwedge xy(x \subseteq \omega \wedge x \notin L \wedge y \subseteq \omega. \rightarrow y \leq_L x)).$$

Sacks' conditions are perfect subsets of $2^{\omega}$ with the above toplogy: it is fruitful to think of a condition $T$ as the set of maximal branches *(paths)* of a tree with one root and no tree tops such that along any one path are infinitely many forks. The forcing relation is then given by

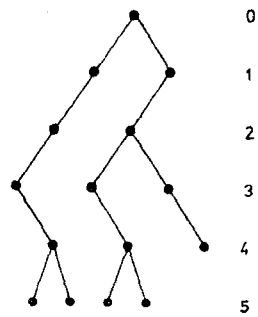



*Fig. 1*

D 1107. $T \Vdash \check{n} \varepsilon \dot{a}$ iff for all paths $P$ in $T$, $P(n) = 1$; $T \Vdash \check{n} \varepsilon \dot{a}$ iff for all paths P in $T$, $P(n) = 0$.

EXAMPLE. Let the initial portion of $T$ be as in Fig. 1.

Then $T \Vdash \check{1} \varepsilon \dot{a}$ and $T\, Q\, \check{3} \varepsilon \dot{a}$.

The principal property of Sacks forcing is

+ T 1108: THE FUSION LEMMA. Let $S$ be the set of finite sequences of 0's and 1's. Let $(T_s \mid s \in S\}$ be a family of Sacks conditions such that for all $s \in S$, $T_{s0} \cap T_{s1} = \emptyset$; $T_{s0} \subseteq T_s$ and $T_{s1} \subseteq T_s$. Then $\bigcap_{n \in \omega} \bigcup_{|s|=n} T_s$ is a Sacks condition.

D 1109 Let $M$ be a c.s.m. of ZF + $V = L$. A real $a$ generic over $M$ with respect to Sacks forcing will be termed a *Sacks real* (over $M$); the model $M[a]$ a *Sacks model.*

For further details, consult Sacks [*].

Another method of obtaining reals of $L$-degree minimal over $L^M$ is to take as conditions pair $\langle A, B \rangle$ with

$$A \subseteq \omega \wedge B \subseteq \omega \wedge A \cap B = \emptyset \wedge \omega\ (A \cup B) = \omega;$$

where

D 1110. $\langle A, B \rangle \Vdash \check{n} \varepsilon \dot{a}$ iff $n \in A$,

$\langle A, B \rangle \Vdash \check{n} \epsilon \dot{a}$ iff $n \in B$.

+ D 1111. We shall call a real a generic with respect to this notion of forcing a *Silver real* (over $M$), and the model $M[a]$ a *Silver model,* in honour of the

man who first established that this forcing relation gives reals of minimal $L$-degree.

Let us note some properties of Silver and Sacks reals over a fixed $M(\Vdash V = L)$.

T 1112. Every non-constructible real in a Sacks model is a Sacks real.

+ T 1113 (KUNEN). No silver real occurs in a Sacks model.

By the minimal degree property, it follows that

T 1114 (KUNEN). No Sacks real occurs in a Silver model.

T 1115 (KUNEN). Not every non-constructible real in a Silver model is a Silver real.

T 1116 (PŘÍKRY). Sacks forcing is equivalent to taking as truth algebra the completion of the algebra of Borel subsets of $2^\omega$ modulo the countable sets.

+ D 1117. The function $f: S(\omega) \to S(\omega) \times S(\omega)$ defined by $f(a) = \langle f_1(a), f_2(a)\rangle$, where

$$n \in f_1(a) \text{ iff } 2n \in a$$

$$n \in f_2(a) \text{ iff } 2n + 1 \in a,$$

is said to *split* the real $a$.

Now if we split a random (or generic) real, we get two random (or generic) reals of incomparable $L$-degrees, but if $a$ is Silver or Sacks, $f_1(a)$ and $f_2(a)$ have comparable $L$-degrees. Further, as Solovay has shown, there are no minimal $L$-degrees in $M^{\mathbf{B}_1}/F_1$ or $M^{\mathbf{B}_2}/F_2$, and if $N$ is a c.s.m. of ZF + GCH + "there are precisely two $L$-degrees", then $N$ contains no reals random or generic over $L^N$. Thus

T 1118. Let $M$ be a c.s.m. of ZF + $V = L$. Then the properties of a real number being generic, random, Silver or Sacks over $M$ are mutually exclusive; moreover if a is a real having any one of these properties, $M[a]$ contains no real having any of the other three.

Silver and Sacks' forcing have another interesting property; which we state in the language of Boolean algebras. A useful theorem in making Boolean valued extensions of $L$ is that if $\mathbf{B}$ satisfies the countable chain condition, then cardinals are preserved in the extension $L^{(\mathbf{B})}$. Now both Silver and Sacks' forcing preserve cardinals, but neither satisfies the countable chain condition.

Let $K$ denote the universe of hereditarily ordinally definable sets. (The paper of MYHILL and SCOTT [*] gives a detailed account of this notion.)

T 1119. $\mathrm{ZF} \vdash V = K + \mathrm{AC}$.

T 1120. $\mathrm{ZF} \vdash L \subseteq K \subseteq V$.

T 1121 (LÉVY). $\mathrm{Con}(\mathrm{ZF})$ implies $\mathrm{Con}(\mathrm{ZF} + V \neq K + K = L + \mathrm{GCH})$.

In fact there is the general principle:

T 1122 (KRIPKE). Let $\mathbf{B}$ be homogeneous. Then

$$V = L \to [L \equiv K]^{\mathbf{B}} = 1.$$

More generally, the conclusion of T 1122 holds of $\mathbf{B}$ is the direct sum of homogeneous algebras. Thus (see KRIPKE [*]) a counterexample to the conjecture of Sikorski that every complete Boolean algebra is the direct sum of homogeneous algebras is given by an algebra $\mathbf{B}$ (which may readily be constructed using the methods of Easton (T 1202)) such that $L^{\mathbf{B}}$

$$[\{n | 2^{\aleph_n} = \aleph_{n+1}\} \varepsilon L]^{\mathbf{B}} = 1.$$

T 1122 and its corollary have been strengthened:

T 1123 (VOPĚNKA). In a Boolean-valued model $V^{(\mathbf{B})}$, the class $K$ is precisely $V^{(\mathbf{B}^*)}$, where

$$\mathbf{B}^* = \{b \in \mathbf{B} \mid b \text{ is fixed under all automorphisms of } \mathbf{B}\}.$$

Solovay remarked, after hearing of Kripke's work, that if $\mathbf{B}$ is the algebra used in proving

T 1124 (MCALOON). $\mathrm{Con}(\mathrm{ZF})$ implies $\mathrm{Con}(\mathrm{ZF} + \mathrm{GCH} + V = K + K \neq L)$.

$\mathbf{B}$ has no automorphisms, and moreover, if $b_1$, $b_2$ are two distinct elements of $\mathbf{B}$, the complete Boolean algebras $\{b \mid b \leq b_1\}$ and $\{b \mid b \leq b_2\}$ are not isomorphic.

T 1125 (MCALOON). $\mathrm{Con}(\mathrm{ZF})$ implies $\mathrm{Con}(\mathrm{ZF} + \mathrm{GCH} + V \neq K + K \neq L)$.

Some brief comments on the proof of T 1124. Suppose that $N$ is a c.s.m. of $\mathrm{ZF} + \mathrm{GCH} + V \neq L$, and that $t \subseteq \omega$, $t \notin L$, $t \in N$. McAloon's original way of making $t$ an ordinal definable was to expand $N$ to an $N'$ in which $t = \{n | 2_{\aleph_n} = \aleph_{n+1}\}$; but then $N'$ does not satisfy GCH. A more subtle method had to be found: the answer is now described.

D 1126. Write $F(\aleph_\alpha)$ for "For every subset $A$ of $\aleph_\alpha$ of cardinality $\aleph_\alpha$ there is a constructible $B \subseteq \aleph_\alpha$, of cardinality $\aleph_\alpha$ with either $B \subseteq A$ or $B \cap A = 0$."

T 1127 (McALOON). Let $M$ be a c.s.m. of ZF + GCH + $\Lambda\alpha > \mathrm{OF}(\aleph_\alpha)$; let $t \in M$, $t \subseteq \omega$, $M \vdash \dot{t} \varepsilon L$. Then there is an extension $N$ of $M$ such that $0n^N = 0n^M$; $N \vdash \mathrm{GCH} \wedge \Lambda\alpha \subseteq \omega F(\aleph_\alpha)$; and in $N$,

$$t = \{n | F(\aleph_{n+1})\}.$$

We could code $t$ higher up the ordinals; for if $\beta$ is any ordinal, the set $\{n|F(\aleph_{\beta+n})\}$ is also ordinal definable. By a judicious iteration of the coding process T 1127, we get a model of ZF + GCH + $V = K$. See McALOON [*].

We end this part some remarks on the property $F$ and en extremely fruitful coding device due to Solovay.

If $M$ is a c.s.m. of ZF + $V = L$, and $t$ is generic, then in $M[t]$, $F(\aleph_0)$ is alse, but for every $\alpha > 0$, $F(\aleph_\alpha)$ is true. (This is the basic fact used in proving T 1127.) In a Sacks model, $F(\aleph_0)$ is true and $F(\aleph_\alpha)$ also holds for every $\alpha \geq 2$.

+ P 1128 (FRIEDMAN). Is $F(\aleph_1)$ true in a Sacks model?

If the answer is yes, then some special properties of Sacks' model must be used in the proof in view of

T 1129. Con(ZF) implies Con(ZF + $\forall a(a \subseteq \omega \wedge V = L[a])$ + $\neg F(\aleph_1)$),

which is a consequence of the following general method of Solovay, hereinafter called "Solovay's trick":

+ T 1130. Let $M$ be a c.s.m. of ZF + AC + $\omega_1^L = \omega_1$, and let $A \in M$, $A \subseteq \omega_1$. Then there is an $t \subseteq \omega$ such that in $M[t]$, "$A \varepsilon \dot{L}[t]$" is true: in other words $A$ is constructible from $t$.

PROOF of T 1130. Let $\langle B_\nu | \nu < \omega_1 \rangle$ be a constructible family of reals, in $M$, s.t. $\nu < \mu < \omega_1 \to B_\nu \cap B_\mu < \omega$, and each $B_\nu$ is infinite. We aspire to find a real $t$ for which

$$\nu(\nu < \omega_1 \to [\nu \in A \leftrightarrow B_\nu \cap t < \omega]).$$

A condition will be a pair $\langle X, Y \rangle$ where $X$ is a finite subset of $\omega$ and $Y$ is a finite subset of $A$, the intended interpretation being

$$t \cap \bigcup_{\nu \in Y} B_\nu = X \cap \bigcup_{\nu \in Y} B_\nu.$$

We therefore define

D 1131. $\langle X, Y \rangle \Vdash \check{n} \varepsilon \dot{t}$ iff $n \in X \cap \bigcup_{\nu \in Y} B_\nu$;

$\langle X, Y \rangle \Vdash \check{n} \varepsilon \dot{t}$ iff $n \in (\bigcup_{\nu \in Y} B) \setminus X$.

The partial ordering of conditions is given by

D 1132. $\langle X, Y \rangle \leq \langle X', Y' \rangle$ iff

$$X \subseteq X \wedge Y' \subseteq Y \wedge X \setminus X' \subseteq \omega \setminus \bigcup \{B_\nu | \nu \in Y\},$$

(where "$\leq$" is read "is a stronger condition than", in sympathy with the Boolean algebraic approach to forcing).

For any $Y$, $Y'$, $\langle X, Y\rangle$ and $\langle X, Y'\rangle$ are compatible, so the countable chain condition is easily shown to hold; the remainder of the proof is elementary.

To prove T 1129, let $N$ be a c.s.m. of ZF + $V = L$; $A$ a subset of $\omega_1^N$ generic with respect to the regular open algebra of $2^{\omega_1}$ with the $\omega$ topology. and let $t$ be as given by T 1130, with $M = N[A]$. Then $F(\aleph_1)$ is false in $M[t]$.

## § 2. *Doing without the Continuum Hypothesis*

We now consider ways and means of violating the GCH while preserving AC.

T 1200 (COHEN). Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} = \aleph_2$).

It is known that $2^{\aleph_0}$ cannot be $\aleph_\omega$; more generally,

T 1201 (J. KÖNIG). ZF + AC $\vdash \bigwedge \alpha[\mathrm{cf}.(\aleph_\alpha) < \mathrm{cf}(2_{\aleph_\alpha})]$.

The following theorem shows that for regular $\aleph_\alpha$, T 1201 is the only restriction.

T 1202 (EASTON). Let $M$ be a c.s.m. of NBG + GCH, and let $F$ be any function (in $M$) with domain $O_n^M$ and range $\subseteq O_n^M$ such that

$$\bigwedge \alpha(\alpha < F'\alpha);\ \bigwedge \alpha\beta(\alpha < \beta \to F'\alpha \leq F'\beta);$$

and

$$\bigwedge \alpha(\mathrm{cf}(\aleph) < \mathrm{cf}(\aleph)).$$

Then there is an extension $N$ of $M$ in which $O_n^M = O_n^N$; $\mathrm{card}^M = \mathrm{Card}^N$; $\mathrm{cf}^M = \mathrm{cf}^N$; AC is true and for every regular $\aleph_\alpha$, $2^{\aleph_\alpha} = \aleph_{F'\alpha}$.

Little is known about singular cardinals; for instance,

+ P 1203 (SOLOVAY). Does Con(ZF) imply Con(ZF + AC + $\bigwedge n < \omega(2^{\aleph_n} = \aleph_{n+1}) + 2^{\aleph_\omega} = \aleph_{\omega+2}$)?

P 1204 (SOLOVAY). Con(ZF + AC + $\bigwedge x \bigwedge n < \omega(x \subseteq \aleph_n \to x \in L) + \bigvee y(y \subseteq \aleph_\omega \wedge y \notin L)$)$\aleph$; or rephrased model-theoretically, can one add a subset of $\aleph_\omega$ in an extension of a model of ZF + $V = L$ without adding subsets of $\aleph_n$, any $n$?

These are the simplest cases of the "singular cardinals problem." There is some restriction on the behaviour of $2^{\aleph_\omega}$, for as pointed out by Hechler and Bukovský independently, if $\bigwedge n \geq n_0$, $2^{\aleph_n} = \aleph_{\omega+1}$, then $2^{\aleph_\omega} = 2^{\Sigma\aleph_n} = \pi(e2^{\aleph_n}) \leq (2^{\aleph_{n_0}})^{\aleph_0} = 2^{\aleph_{n_0}} = \aleph_{\omega+1} \leq 2^{\aleph_\omega}$ so $2^{\aleph_\omega} = \aleph_{\omega+1}$.

This remark is a special case of

+ T 1205 (BUKOVSKÝ [H25]). ZF + AC $\vdash$ Let $\mathrm{cf}(\aleph_\alpha) \leq \aleph_\beta < \aleph_\alpha$. Then

(i) $2^{\exp \aleph_\alpha} = \sum_{\gamma \in \alpha} (2^{\exp \aleph_\gamma})$ if there is a $\gamma_0 < \alpha$ such that

$$\bigwedge \gamma(\gamma_0 < \gamma < \alpha \to 2^{\exp \aleph_\gamma} = 2^{\exp \aleph_{\gamma_0}});$$

$$2^{\exp \aleph_\alpha} = \left(\sum_{\gamma<\alpha} (2^{\exp \aleph_\gamma})\right)^{\exp\left(\mathrm{cf}\left(\sum_{\gamma<\alpha} (2^{\exp \aleph_\gamma})\right)\right)}$$

otherwise.

(ii) $\aleph_\alpha \exp \aleph_\beta = \sum_{\gamma<\alpha} (\aleph_\beta \exp \aleph_\gamma)$, if there is a $\gamma_0 < \alpha$ such that for all $\gamma$ with $\gamma_0 < \gamma < \alpha$,

$$\aleph_\gamma \exp \aleph_\beta = \aleph_{\gamma_0} \exp \aleph_\beta;$$

$$\aleph_\alpha \exp \aleph_\beta = \left(\sum_{\gamma<\alpha} (\aleph_\gamma \exp \aleph_\beta)\right)^{\exp\left(\mathrm{cf}\left(\sum_{\gamma<\alpha} (\aleph_\gamma \exp \aleph_\beta)\right)\right)}$$

otherwise.

Thus in a precise sense the function $\lambda\alpha\beta\aleph_\alpha \exp \aleph_\beta$ can be calculated from $\lambda\alpha\aleph_\alpha \exp \mathrm{cf}(\aleph_\alpha)$. Hence the general problem:

P 1206. Investigate the behaviour of $\lambda\alpha\aleph_\alpha \exp \mathrm{cf}(\aleph_\alpha)$ in various models.

+ P 1207. Do there exist two models $M \subseteq N$ of ZF + AC with the same cardinals but different cofinalities?

PŘÍKRY has shown (T 1218) that if there is a two-valued measurable cardinal $\varkappa$ in $M$, a c.s.m. of ZF + AC, then there is an $N \supseteq M$ with the same cardinals as $M$ in which $\mathrm{cf}^N(\varkappa) = \omega$.

P 1208 (BUKOVSKÝ). Show

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + 2^{\exp \aleph_0} = \aleph_1 + \bigwedge n \geq 1 (2^{\exp \aleph_n} = \aleph_{\omega+2} = 2^{\exp \aleph_{\omega+1}}) +$$

$$+ \aleph_\omega \exp \aleph_0 = \aleph_{\omega+2} + \bigwedge \alpha > \omega (2^{\exp \aleph_\alpha} = \aleph_{\alpha+1}).$$

P 1209 (TAKEUTI). Does Con(ZF) imply

$$\mathrm{Con}\left(\mathrm{ZF} + \mathrm{AC} + \bigwedge n < \omega \bigwedge m < \omega \left(2^{\aleph_n} = 2^{\aleph_m} \wedge L(S(\aleph_n)) = L(S(\aleph_m))\right)\right)?$$

P 1210. Find a real $t$ and a c.s.m. $M$ of ZF + GCH such that

$$M[t] \models \mathrm{ZF} + \mathrm{AC} + 2^{\aleph_0} = \aleph_2$$

and the cardinals of $M[t]$ are those of $M$.

T 1211 (SILVER; SOLOVAY). Let $M$ be a c.s.m. of ZF + $V = L$. Then there is a real $t$ such that $M[t]$ is a model of ZF + AC, and

$$\aleph_1^{M[t]} = \aleph_1^M;\ \aleph_2^{M[t]} = \aleph_2^M \approx \aleph_3^M$$

in $M[t]$.

T 1211 is proved by using Solovay's trick, T 1130, as is

+ T 1212 (JENSEN). Let $M$ be a c.s.m. of ZF + $V = L$. Let $\theta$ be any ordinal strongly inaccessible in $M$. Then there is a $t \subseteq \omega$ such that $M[t] \models$ ZF + GCH and in $M[t]$, $\theta = \aleph_1$.

(Solovay had obtained T 1212 for the special case when $\theta$ is the first inacessible in $M$).

T 1213 (KUNEN). ZF $\vdash$ If there is a c.s.m. of ZF + $V = L$, then there is a c.s.m. $N$ of ZF + AC such that no submodels of $N$ with the same cardinals satisfies GCH.

We have seen McAloon's methods for making sets of ordinals rodinal-definable, and have noted that the we need only use the higher reaches of the ordinals for coding; for example given $t \subseteq \omega$ we can select an $\alpha$ and make

$$t = \{n \mid 2 \exp \aleph_{\alpha+n} = \aleph_{\alpha+n+1}\};$$

by intelligent iteration we may prove

+ T 1214 (MCALOON). Con(ZF) implies

$$\text{Con}(\text{ZF} + \text{AC} + 2^{\aleph_0} = \aleph_{21} + 2^{\aleph_1} = \aleph_{\omega+5} + V + K + K \neq L).$$

+ T 1215 (MCALOON). Con(ZF) implies

$$\text{Con}(\text{ZF} + \text{AC} + 2^{\aleph_0} = \aleph_2 + V \neq K + K \neq L).$$

Throughout 1216–1220, $N$, $M$ are c.s.m's of ZF + AC with the same ordinals, and $N \subseteq M$.

+ D 1216 (VOPĚNKA). A set $Z \in N$, $Z \subseteq M$ is a *support* for $N$ over $M$ iff

$$\bigvee X \in N(X \subseteq M \to \bigvee R \in M(R''Z = X)).$$

T 1217 (VOPĚNKA). A support for $N$ over $M$ exists iff there exists a complete Boolean algebra $\mathbf{B} \in M$ and an $M$-complete ultrahomomorphism $F$ such that $N = M^{\mathbf{B}}/F$.

T 1218 (VOPĚNKA). Let $K$ be the class fo hereditarily ordinal definable sets of $M$. Let $A \in M$. If $L[A] \supseteq K$, then there is a $\mathbf{B} \in K$ and an $F$ such that $L[A] = K^{\mathbf{B}}/F$.

T 1219 (VOPĚNKA). Let $Z \subseteq A \in M$. Then $Z$ is a support over $M$ for some $N$ iff

(1) $$\forall R_1 \in M(R_1'' Z = S^M(Z))$$

and

(2) $$\forall R_2 \in M(R_2'' Z = S^M(A/Z)).$$

We may also treat the notion of a Boolean-valued model when $\mathbf{B}$ is a proper class of $M$. $\mathbf{B}$ is *L-complete* iff it contains suprema and infima of constructible subsets.

T 1220 (VOPĚNKA). Suppose that $M \vdash V = L$ and that G&del's axiom $E$, the class form of the axiom of choice, holds in $N$. Then there is a constructible $L$-complete Boolean alebra $\mathbf{B}$ and an $L$-complete ultrahomomorphism $F$ such that $N = M^{\mathbf{B}}/F$. Moreover, $S(L) = \{r'' F \mid r \in L\}$; that is, $F$ is a support for $N$ over $M$, except that $F$ is not necessarily a net of $N$.

Easton has shown that axiom $E$ is independent of NBG + AC; see T 5002.

Souslin originally formulated his hypothesis as "Every complete total ordering with neither first nor last elements which satisfies the countable chain condition on open intervals in order-isomorphic to the open interval $(0, 1)$": we shall call this form $\mathrm{SH}^1$.

+ D 1221. A *Souslin tree* is a partial ordering $\langle T, <\rangle$ such that for all $x \in T$, $\langle \{y \mid y < x\}, <\rangle$ is a well-ordering, every chain and antichain is countable, but $\overline{\overline{T}} = \aleph_1$.

$\mathrm{SH}^1$ was shown by E. C. Miller to be equivalent (in ZF + AC) to the assertion that there are no Souslin trees, whizh form we shall call $\mathrm{SH}^2$.

D 1222. A *normal* Souslin tree $\langle T, <_T\rangle$ is one which has precisely one minimal element, in which there are infinitely amny branches at each point, whose field $T$ is $\aleph_1$, the set of countable ordinals, whose ordering is such that $\alpha <_T \beta \rightarrow$ $\rightarrow \alpha \in \beta$, and in which

$$\wedge \alpha \overline{\overline{\{\beta \mid \alpha <_T \beta\}}} = \aleph_1$$

D 1223. We write SH for "There are no normal Souslin trees". SH is equivalent is ZF + AC to $\mathrm{SH}^1$ and $\mathrm{SH}^2$.

T 1224 (TENNENBAUM, JECH). Con(ZF) implies Con(ZF + GCH + $\neg$ SH).

T 1225 (TENNENBAUM, JECH). Con (ZF) implies Con(ZF + AC + $\neg$ SH + $+ 2^{\aleph_0} = \aleph_2$).

T 1226 (SOLOVAY, TENNENBAUM). Con(ZF) implies Con(ZF + AC + SH).

We shall first make some remarks about the original proofs of these theorems, and discuss certain generalization of SH to higher cardinals; and then we shall consider an axiom that that implies SH and has other virtues.

T 1224 may be proved by adding a Souslin tree to a c.s.m. $M$ of ZF + + $V = L$: Tennebaum worked with finite conditions, and it was remarked by Solovay that the algebra he used was isomorphic to the one used to add $\aleph_1$ generic reals, viz. the regular open algebra of $2^{\aleph_1}$ with the $\aleph_0$ topology. Another method, developed in Czechoslovakia by JECH [H 50], and modified by Příkrý, is to take as conditions trees which satisfy the requirements of D 1221 and D 1223 save that they are countable. One tree is stronger than another (qua conditions) if it is an end extension of it. Jech mentions in a letter to the author that his algebra is the regular open algebra of $2^{\aleph_1}$ with the $\aleph_1$ topology: thus no new reals are added with his method.

The proof of T 1226 is much harder: suppose there is a normal Souslin tree $T$ in M. We may destroy it by adding a generic path through $T$, a condition being an initial segment of a path through $T$, ending at some point of $T$, and the partial ordering of conditions being that given by inclusion (the countable chain condition follows from the fact that in $T$, all antichains are countable); but new trees may be produced in the act. Now in $M$ there are at most $\aleph_2$ normal Souslin trees. We destroy one; then in the extansion there are still $\leq \aleph_2$ trees, and as cardinals are preserved, it is the same $\aleph_2$. We now iterate the process, zigzagging back and forth so that every normal Souslin tree in the sequence of models, which will be of length $\aleph_2$ is destroyed precisely once. Then provided that no cardinals have collapsed in the meantime, SH will be true in the final model. The original demonstration has been considerable simplified by the following general principle of iteration:

T 1227 (SOLOVAY). ZF + AC $\vdash$ Let $\varkappa$ be an infinite ordinal; let $\langle \mathbf{B}_\gamma \mid \nu \leq \varkappa \rangle$ be a sequence of complete Boolean algebras such that $B_0 = 2$; each $\mathbf{B}_\gamma$ is a regular subalgebra of $\mathbf{B}_{\gamma+1}$; for $\lambda$ a limit ordinal $\leq \lambda$, $\mathbf{B}_\lambda$ = the minimal completion of $\bigcup_{\gamma<\lambda} \mathbf{B}_\lambda$; then the first $\nu$ such that $\mathbf{B}_\gamma$ does not satisfy the countable chain condition is not a limit ordinal.

But after $\aleph_2$ iterations we find that possibly $2^{\aleph_0} > \aleph_1$; thus the main problem now is this:

+ P 1228. Does Con(ZF) imply Con(ZF + GCH + SH)ℵ

+ P 1229. Is SH true in $L$ℵ

+ P 1230 (SOLOVAY). If one extends a c.s.m. of ZF + $V = L$ by adding $\aleph_1$ random reals, is SH necessarily false in the extensionℵ

In T 1226 using Easton's method, one can get $2^{\aleph_0} = 2^{\aleph_1} =$ anything permitted by T 1201, but it is not known how to get $2^{\aleph_0} < 2^{\aleph_1}$. The function $\langle 2^{\aleph_\alpha} \mid \alpha > 1 \rangle$ in T 1226 and the function $\langle 2^{\aleph_\alpha} \mid \alpha < 0 \rangle$ in T 1225 can be made to do anything sensible.

T 1231 (RUDIN [8]). ZF + AC $\vdash$ If every normal Hausdorff space is countably paracompact, then SH.

+ P 1232 (TENNENBAUM). Prove that Con(ZF) implies Con(ZF + GCH + every normal Hausdorff space is countably paracompact).

D 1233. Let $\varkappa$ be an infinite cardinal. A *Souslin $\varkappa$-tree* is a tree of power $\varkappa$ such that every chain and antichain is of power $< \varkappa$.

+ T 1234. ZF + AC $\vdash$ If $\varkappa$ is weakly compact, then there are no Souslin $\varkappa$-trees.

T 1235 (HRBÁČEK [H51]). Let $\varkappa$ be a regular infinite cardinal in a c.s.m. $M$ of ZF + $V = L$. In $M$, let $\mathbf{B}$ be the regular open algebra of $2^{\gamma^+}$ with the $\varkappa^+$ topology. Then

$$[\text{There is a Souslin } \hat{\varkappa}^+\text{-tree}]^{\mathbf{B}} = \mathbf{1}.$$

Jech's method is the case $\varkappa = \omega$. Příkrý has shown tha the method works also for adding a $\varkappa$-tree where $\varkappa$ is inaccessible or Mahlo (and preserving these properties in the extension); Silver has discovered how to add a $\varkappa^+$-tree if $\varkappa$ is a singular cardinal.

Several relative consistency results have been obtained by iteration techniques similar to the one just described: in each case, a series of extensions of $L$ is made, each adding an object generic with respect to some collection of subsets of a complete Boolean algebra. A number of people (Kunen; Martin; Rowbottom; Silver) incependently realized that much of the pain of iteration arguments could be removed by working with an axiom asserting that the required generic objects exist in the real (and not a Boolean-valued) universe. We select one possible formulation of such a principle: Solovay and Martin intend to write a paper about Martin's version to be entitled "Internal Cohen Extensions".

+ D 1236. We write [*] for the following assertion: for every complete Boolean algebra $\mathbf{B}$ satisfying the countable chain condition adn every family of cardinality at most $\aleph_1$ of subserts of $\mathbf{B}$, there is a homomorphism of $\mathbf{B}$ onto 2 complete with respect to every set in the family.

The following result can be proved by iterated forcing:

+ T 1237. Con(ZF) implies Con(ZF + AC + (*)).

$+$ T 1238. ZF $+$ AC $+$ (*) $\vdash 2^{\gamma_0} > \aleph_1$. Both $2^{\gamma_0} = \aleph_2$ and $2^{\gamma_0} < \aleph_2$ are consistent relative to ZF $+$ AF $+$ (*).

T 1226 may now be obtained as a corollary of T 1237 and

T 1239. ZF $+$ AC $+$ (*) $\vdash$ SH.

$+$ T 1240 (Solovay). ZF $+$ AC $+$ (*) $\vdash$ Every subset of $\aleph_1$ is constructible from a real.

T 1239 is readily seen: If $T$ were a normal Souslin tree, the Tennenbaum destruction algebra described in the discussion of the proof of T 1226 would be a counterexample to (*). T 1240 uses T 1130.

Further consequences of (*) will be given in Section 3: we note here some topological applications.

$+$ T 1241 (Silver). ZF $+$ AC $+$ (*) $\vdash$ There is a non-metrizable separable normal Moore space.

(For Moore spaces, see Bing [1].)

$+$ T 1242 (Booth). ZF $+$ AC $+$ (*) $\vdash 2^{\omega_1}$ is sequentially compact, where 2 has the discrete topology and $2^{\omega_1}$ the product topology.

In constrast to 1242, $2^c$ cannot be sequentially compact, where $c = 2^{\gamma_0}$.

T 1243 (Booth). ZF $+$ AC $+$ (*) $\vdash$ No compact separable metric space of positive dimension is the union of $\aleph_1$ closed sets of dimension 0.

T 1244 (Bukovský [H55]). Con(ZF) implies Con(ZF $+$ AC $+ 2^{\gamma_0} = 2^{\gamma_1} +$ every uncountable separable metric space contains a non-Borel set).

$+$ § 3. *Life Without Choice* "$C_n^{wo}$"

We use Lévy's notation [5] for various weak forms of the axiom of choice, though we abbreviate "$\bigwedge\alpha C_n^{\alpha}$" by "$C_n^{wo}$". By $DC^{\alpha}$ (where $\alpha$ is finite or an aleph) we mean the following:

given a relation $R$ such that for every subset $Y$ of a set $X$ with $\overline{\overline{Y}} < \alpha$ there is an $x \in X$ with $(Y, x) \in R$, there is a function $f: \alpha \to X$ such that $\bigwedge\beta < \alpha \langle f''\beta, f'\beta\rangle \in R$.

Thus $DC^{\omega}$ is the axiom of dependent choices.

[5] gives a thorough treatment of the Fraenkel–Mostowski family of results, some remarks on which will be found in a later section of this saga.

T 1300 (Cohen). Con(ZF) implies Con(ZF $+ \neg$ AC).

T 1301 (Cohen). Con(ZF) implies Con(ZF $+ \neg C_2^{\omega}$).

T 1302 (JENSEN). Con(ZF) implies Con(ZF + $C^{\omega}$ + $\neg DC^{\omega}$).

1302 can be strengthened to

+ T 1303 (JENSEN). Let $\alpha$ be an infinite cardinal in a c.s m. $M$ of ZF + $V = L$; then $M$ can be extended to an $N$ in which $O_n^M = O_n^N$; $\mathrm{Card}^M = \mathrm{Card}^N$; $C^{\alpha}$ is true but $DC^{\omega}$ is false.

On the other hand,

+ T 1304 (JENSEN). ZF $\vdash \Lambda\alpha C^{\alpha} \rightarrow DC^{\omega}$.

T 1305. ZF $\vdash \Lambda\alpha DC^{\alpha} \rightarrow AC$.

T 1306 (SOLOVAY, based on work of FEFERMAN and JENSEN). Con (ZF) implies Con(ZF + $\Lambda\alpha^{\alpha}$ + $\neg DC^{\omega_1}$).

T 1307 (JECH [H44]). ZF $\vdash$ Let $M$ be a c.s.m. for ZF + AC, and $\alpha$ a regular infinite cardinal in $M$. Then there is an $N \supseteq M$ with the same ordinals such that in $N$, $\neg DC^{\alpha}$, $\neg C_2^{\alpha}$ and there is a cardinal incomparable with $\alpha$, but for every $\beta < \alpha$, $DC^{\beta}$, $C^{\beta}$ and $\beta$ is comparable with every cardinal.

T 1308 (DERRICK, DRAKE; HÁJEK, VOPĚNKA [H31]). Let $\lambda \in O_n^M$, where $M$ is a c.s.m. of ZF + $V = L$, be an infinite ordinal not the successor of a limit ordinal; then there is an $N \supseteq M$ with the same ordinals and in which $V_{\lambda}$ is well ordered but $V_{\lambda+1}$ is not.

For a detailed discussion of the finite axioms of choice $C_n$, the reader is referred to the thesis of ZUCKERMAN; see [*]. The present test problems

+ P 1309 (MOSTOWSKI). Does ZF $\vdash C_3 \wedge C_5 \wedge C_{13} \rightarrow C_{15}$?

Gauntt has shown, though, that

T 1310. ZF $\vdash \Lambda m, n < \omega[C_{m+n} \rightarrow C_m$ or $C_n^{wo}]$,

and that consequently

T 1311 (GAUNTT). ZF $\vdash C_3 \wedge C_5 \wedge C_{13} \rightarrow C_{15}^{wo}$.

P 1309 would be true if the following were:

P 1312. ZF $\vdash C_{m+n} \rightarrow C_m$ *or* $C_n$.

P 1313. Find a finitary procedure for deciding of any $n, n_1, \ldots, n_k$, whether ZF $\vdash C_{n_1} \wedge \ldots \wedge C_{n_k} \rightarrow C_n$.

Gauntt has a partial answer to P 1313 for the system ZFU, which is ZF weakened to admit Urelemente: it is conjectured that his work will transfer to ZF. His results are revealed in Section 5. We now discuss the relationships of sundry consequences of AC. Some definitions: we write

D 1314. $O$ for "Every set can be totally ordered"

OE for "Every partial ordering can be extended to a total ordering"

BPI for "Every Boolean algebra has a prime ideal"

$S$ for "Every homomorphism of a subalgebra of a Boolean algebra **A** into a complete Boolean algebra **B** can be extended to an homomorphism of **A** into **B**"

T 1315 (SIKORSKI). ZF $\vdash$ AC $\rightarrow S$.

T 1316 (LUXEMBURG). ZF $\vdash S \rightarrow$ BPI.

T 1317 ZF $\vdash$ PFI $\rightarrow$ OE (via the compactness theorem) . $\wedge$ .OE $\rightarrow O$.

T 1318 (COHEN). Con(ZF) implies Con(ZF + $\neg O$).

+ T 1319 (MATHIAS). Con(ZF) implies Con(ZF + $O$ + $\neg$ OE).

+ T 1320 (HALPERN, LÄUCHLI, LÉVY). Con(ZF) implies Con(ZF + BPI + + $\neg$ AC).

+ P 1321. Does Con(ZF) imply Con(ZF + OE + $\neg$ BPI)?

P 1322. Does Con(ZF) imply Con(ZF + PBI + $\neg S$)?

P 1323. Does Con(ZF) imply Con(ZF + $S$ + $\neg$ AC)?

+ P 1324. Does Con(ZF) imply Con(ZF + BPI + $DC^{\omega}$ + $\neg$AC)?

Felgner has proved in ZF without the axiom of foundations that if every total ordering has a cofinal sub-well ordering, then the power set of a well-ordered set is well-orderable; as a corollary,

+ T 1325 (FELGNER). ZF $\vdash$ If every total ordering has a cofinal subwell-ordering, then AC.

T 1326. ZF $\vdash$ PBI $\rightarrow$ the Hahn–Banach theorem.

+ P 1327. Is BPI probable in ZF + the Hahn–Banach theorem?

T 1328 (SOLOVAY). Con(ZF) implies Con(ZF + $DC^{\omega}$ + every endomorphism of the additive subgroup of the real line is continuous).

T 1329 (LÉVY). Con(ZF) implies Con(ZF + $cf(\omega_1) = \omega$ + the real line is the countable union of countable sets + $\neg\, \Sigma^1_3$ – AC)

+ P 1330. Find a model $M$ such that no extension with the same ordinals satisfies AC.

Even better,

+ P 1331. Find a model in which $\bigwedge \alpha \in On(cf(\aleph_\alpha) = \omega)$.

We shall now use German letters $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ to denote cardinals (not necessarily alephs).

+ P 1332. Does $\bigwedge \mathfrak{m}(\mathfrak{m} \geq \omega \to \mathfrak{m} + \mathfrak{m} = \mathfrak{m})$ inply AC in ZF?

For other problems of this sort, see Lévy [5].

D 1333. $\mathfrak{m}$ adj $\mathfrak{n}$ iff $\mathfrak{m} < \mathfrak{n} \wedge \neg \bigvee \mathfrak{p}[\mathfrak{m} < \mathfrak{p} < \mathfrak{n}]$.

T 1334 (Specker). $\mathrm{ZF} \vdash \neg \bigvee \mathfrak{m}[2^{\mathfrak{m}} \leq \mathfrak{m}^2]$;

hence

T 1335 (Specker; Lindenbaum, Tarski). $\mathrm{ZF} \vdash \bigwedge \mathfrak{m}[\mathfrak{m}$ adj $2^{\mathfrak{m}}$ adj $2^{2^{\mathfrak{m}}} \to 2^{\mathfrak{m}}$ an aleph].

P 1336. Does Con(ZF) imply Con(ZF + $\bigvee \mathfrak{m}[\mathfrak{m}^2 \not\leq 2^{\mathfrak{m}}]$)?

P 1337. Does $\mathrm{ZF} \vdash \mathfrak{m}$ adj $2^{\mathfrak{m}} \to \mathfrak{m}$ is an aleph?

T 1338 (Solovay). Con(ZF + $V = L$ there is an inaccessible cardinal) implies Con(ZF + $\bigvee \mathfrak{m}(\mathfrak{m}$ adj $2^{\mathfrak{m}}$ and $2^{\mathfrak{m}}$ not an aleph)).

P 1339. Is the conclusion of 1338 deducible from Con(ZF)?

P 1340 (Conway). Does Con(ZF) imply

$$\mathrm{Con}(\mathrm{ZF} + \neg \mathrm{AC} + \bigwedge \mathfrak{m} \bigwedge \mathfrak{n}(\mathfrak{m}\mathfrak{n} = \mathfrak{m} + \mathfrak{n} \to \mathfrak{m} \leq \mathfrak{n} \text{ or } \mathfrak{n} \leq \mathfrak{m}))?$$

D 1341 (Hartogs). $\aleph(\mathfrak{m}) =$ the least ordinal $\alpha$ s.t. $\overline{\overline{\alpha}} \not\leq \mathfrak{m}$.

T 1342. $\mathrm{ZF} + \mathrm{YC} \vdash \aleph(2^{\aleph_0}) \neq \aleph_\omega$.

T 1343 (Derrick, Drake). Con(ZF) implies Con(ZF + $\aleph(2^{\aleph_0}) = \aleph_\omega$). In fact, given a c.s.m. $M$ of ZF + GCH, and an $\alpha > 0$, $\alpha \in O_n^M$, there is and $N \supseteq M$ with $\mathrm{Card}^N = \mathrm{Card}^M$ and $N \models \mathrm{ZF} + \aleph(2^{\aleph_0}) = \aleph_\alpha$.

T 1344. $\mathrm{ZF} \vdash \mathfrak{m}$ adj $2^{\mathfrak{m}} \wedge 2^{\mathfrak{m}}$ not an aleph $\to \aleph(\mathfrak{m})$ is a limit cardinal in $L$ and $\aleph(\mathfrak{m}) = \aleph(2^{\mathfrak{m}}) \leq * 2^{\mathfrak{m}}$.

In the model constructed by Lévy to prove T 1329, $\aleph(2^{\aleph_0}) = \aleph_1$.

P 1345 (Hájek). In Lévy's model, $\neg$ ($\aleph_0$ adj $2^{\aleph_0}$).

By T 1344. $2^{\aleph_0}$ is not adjacent to 2 exp $2^{\aleph_0}$ in Lévy's model.

D 1346. DF($\mathfrak{m}$) for $\aleph(\mathfrak{m}) = \aleph_0$ ("Dedekind finite").

P 1347. Does Con(ZF) imply

$$\mathrm{Con}(\mathrm{ZF} + \bigvee \mathfrak{m}(\mathrm{DF}(\mathfrak{m})) + \bigwedge \mathfrak{m} \bigwedge \mathfrak{n}(\mathrm{DF}(\mathfrak{m}) \wedge \mathrm{DF}(\mathfrak{n}) \to . \mathfrak{m} < \mathfrak{n} \text{ or } \mathfrak{n} < \mathfrak{m}))?$$

The paper of Ellentuck [2] describes researches into the theory of Dedekind finite cardinals.

Other results for theories negating the axiom of choice are described in Section 3.

T 1348 (DERRICK, DRAKE). Con(ZF) implies

$$\mathrm{Con}(\mathrm{ZF} + \neg\,\mathrm{AC} + \bigwedge\alpha(\aleph(2^{\aleph_\alpha}) = \aleph_{\alpha+2})).$$

P 1349. Does $\mathrm{ZF} \vdash \neg\bigwedge\alpha(\aleph(2^{\aleph_\alpha}) = \aleph_{\alpha+1})$?

We end with a theorem about the possible patterns of cardinals, and an answer to a question of Tarski.

T 1350 (JECH [H36]). ZF $\vdash$ Let $M$ be a c.s.m. for ZF + AC; $\langle A, \prec\rangle$ a partially ordered set in $M$. Then there is an $N \supseteq M$ with the same ordinals and a set $B \in N$ of cardinals such that

$$\langle A, \prec\rangle \cong \langle B, \leq\rangle.$$

P 1351. Under what conditions on $\langle A, \prec\rangle$ is there a $B \in N$, such that

$$\langle A, \prec\rangle \cong \langle C, \leq\rangle$$

where $C$ is the set of cardinals of subsets of $B$?

D 1352 (TARSKI). $\mathfrak{m}$ is a strong successor of $\mathfrak{n}$ iff $\mathfrak{m} > \mathfrak{n} \wedge \mathfrak{p}(\mathfrak{p} < \mathfrak{m} \to \mathfrak{p} \leq \mathfrak{n})$.

T 1353 (JECH [H40]). Con(ZF) implies Con(ZF + $\bigvee\mathfrak{m}$($\mathfrak{m}$ has no strong successor)).

+ P 1354. Does ZF + "Every cardinal has a strong successor" $\vdash$ AC?

## § 4. *Proving two-quantifier statements*

Lévy has classified the sentences AC, GCH, $V = L$ in his hierarchy [6]

T 1400 (LÉVY). AC, GCH, $V = L$ are all $\Pi_2$-sentences. Let $\Phi \in \Sigma_2$.

T 1401 (LÉVY). If ZF + GCH $\vdash \Phi \to V = L$ then ZF + GCH $\vdash \neg\,\Phi$.

T 1402 (LÉVY). If ZF + AC $\vdash \Phi \to$ GCH then ZF + AC $\vdash \neg\,\Phi$.

T 1403 (LÉVY). If ZF $\vdash \Phi \to$ AC then ZF $\vdash \neg\,\Phi$.

T 1404 (LÉVY). If ZF $\vdash \Phi \to C_2^\omega$, then ZF + AC $\vdash \neg\,\Phi$.

The proofs are described in LÉVY' lecture [*].

The predicate "$x = 2^{\aleph_\alpha}$" is $\Pi_2$: we may consider characterizing cardinals $\varkappa$ by properties expressible in higher order logic. Let $\Sigma_n^1 \ldots$ refer to this hierarchy. Then the cardinal $2^{\aleph_0}$ can be characterized by the $\Sigma_2^1$ property that it admits relations giving it the structure of a complete ordered field.

1405 (KUNEN). Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} < \aleph_{\omega_1}$ + the cardinal $2^{\aleph_0}$ is not $\Pi^1_2$ characterizable).

The lecture of KUNEN [*] gives a full discussion.

## Section 2. Large cardinals

### § 0. *Measurability and Compactness*

D 2000. $\varkappa$ is 2VM iff $\varkappa$ is a 2-valued measurable cardinal.

$\varkappa$ is RVM iff $\varkappa$ real-valued but not 2-valued measurable.

$\varkappa$ is $A$-Mahlo (where $A \subseteq \varkappa$) iff every nonempty closed unbounded subset of $\varkappa$ contains a member of $A$ and $\varkappa$ is weakly inaccessible.

If $A = \{\alpha < \varkappa \mid \alpha \text{ is regular}\}$, $\{\alpha < \varkappa \mid \alpha \text{ is weakly inaccessible}\}$, $\{\alpha < \varkappa \mid \alpha$ is strongly inaccessible$\}$ then $\varkappa$ is Mahlo, weakly Mahlo, strongly Mahlo, respectively.

$i(\varkappa, \lambda)$ iff $\varkappa$, $\lambda$ are cardinals and $\varkappa$ carries a $\lambda$-saturated $\varkappa$-complete nontrivial ideal.

An ideal $I$ on $\varkappa$ is *normal* iff for every function $f: \varkappa \to \varkappa$ such that $\{\alpha \mid f(\alpha) < \alpha\} \notin I$, there is a $\beta < \varkappa$ with $\{\alpha \mid f(\alpha) = \beta\} \notin I$.

A filter on $\varkappa$ is *normal* iff its dual ideal is.

A measure on $\varkappa$ is *normal* iff the ideal of sets of measure 0 is.

The following subsumes all previously popular partition properties.

+ D 2001 (MOSCHOVAKIS). Let $\varkappa$, $\lambda$, $\nu$ be cardinals, $\mu$ an ordinal. $\varkappa \vec{\lambda}(\mu)^{<\omega}_{\nu}$ iff for every $f$ with domain the set of sequences of length $\lambda$ of finite subsets of $\varkappa$ and range a subset of $\nu$ (in symbols, $f: ([\varkappa]^{<\omega})^{\lambda} \to \nu$), there is a sequence $\langle x_\mu \mid \eta < \lambda\rangle$ of subsets of $\varkappa$, each of order type $\mu$, (i.e., $\langle x_\mu, \in \ x_\mu\rangle \cong \mu$), with the property that for every $s$ and $t$ in the domain of $f$, if for all $\eta < \lambda$, $s_\mu \subseteq x_\mu$, $t_\mu \subseteq x_\mu$, and $\bar{s}_\mu = \bar{t}_\mu$, then $f(s) = f(t)$.

The case $\lambda = 1$ is the Erdős–Rado property $\varkappa \to (\mu)^{<\omega}_{\nu}$. The case $\nu = 2$ will be written $\varkappa\, \vec{\lambda} \to (\mu)^{<\omega}$.

$\varkappa$ is a *Ramsey cardinal* iff $\varkappa \to (\varkappa)^{<\omega}$, $\varkappa$ is a *Moschovakis cardinal* iff $k_{\omega_1} \to (\omega_1)^{<\omega}$.

D 2002. We write MC for $V\varkappa(\varkappa$ is 2VM),

RMC for $V\varkappa(\varkappa$ is RVM),

SCC for $V\varkappa(\varkappa$ is strongly compact),

SuCC for $V\varkappa(\varkappa$ is supercompact).

The next theorems list the basic relations between these properties.

+ T 2003. ZF + AC $\vdash \kappa$ supercompact $\rightarrow \kappa$ strongly compact;

$$\kappa \text{ strongly compact} \rightarrow \kappa \text{ 2VM};$$

$$\kappa \text{ 2VM} \rightarrow \kappa \text{ Ramsey};$$

$$\beta < \alpha \leq \kappa \wedge \kappa \rightarrow (\alpha)^{<\omega}. \rightarrow \kappa \rightarrow (\beta)^{<\omega};$$

$$i(\kappa, \lambda) \wedge \lambda' > \lambda. \rightarrow i(\kappa, \lambda');$$

$$\kappa \text{ RVM} \rightarrow i(\kappa, \omega_1);$$

$$i(\kappa, \lambda^+) \rightarrow \kappa \text{ is 2VM or } \kappa \leq 2^\lambda;$$

$$\kappa \text{ 2VM iff } i(\kappa, 2);$$

$\kappa$ is weakly compact and strongly inaccessible iff $\kappa$ is $\Pi^1_1$-indescribable;

$$\kappa \text{ Ramsey} \rightarrow \kappa \ \Pi^1_1\text{-indescribable};$$

$$\kappa \ \mathrm{H}^1\text{-indescribable} \rightarrow \text{Mahlo};$$

$$\kappa \text{ 2VM} \rightarrow \kappa \ \Pi^1_1\text{-indescribable}.$$

D 2004. $\kappa(\alpha) =$ the least $\lambda$ s.t. $\lambda \rightarrow (\alpha)^{<\omega}$, if one exists.

T 2005 (Rowbottom). ZF + AC $\vdash \lim(\beta) \ \forall\lambda(\lambda \rightarrow (\beta)^{<\omega}) \wedge \alpha < \beta. \rightarrow \kappa(\alpha) < \kappa(\beta)$.

T 2006 (Silver). ZF + AC $\vdash$ If $\kappa(\omega)$ exists then there is a $\lambda < \kappa(\omega)$ which is $\Pi^1$-indescribable for all $m$ and $n$; $\kappa(\omega)$ itself is $\Pi^1_1$-describable.

+ T 2007 (Erdős). ZF + AC $\vdash$ If $\alpha$ is a limit ordinal $\forall\lambda(\lambda \rightarrow (\alpha)^{<\omega})$ then $\kappa(\alpha)$ is strongly inacessible.

T 2008 (Silver). ZF + AC $\vdash$ If $\lim(\alpha)$ and $\forall\lambda(\lambda \rightarrow (\alpha)^{<\omega})$, then for all cardinals $\nu < \kappa(\alpha)$, $\kappa(\alpha) \rightarrow (\alpha)^{<\omega}_\nu$. In particular, if $\alpha$ is Ramsey (i.e., if $\alpha = \kappa(\alpha)$), then $\alpha \rightarrow (\alpha)^{<\omega}_\nu$ for all $\nu < \alpha$.

T 2009. ZF + AC $\vdash$ If $\kappa$ is 2VM, there is a $\lambda < \kappa$ which is Ramsey.

+ T 2010. ZF + AC $\vdash$ If $\kappa$ is weakly compact, there is a $\lambda < \kappa$ which is Mahlo.

Little is known about the size of Moschovakis cardinals relative to other large numbers; Silver has made the following simple observation:

+ T 2011. ZF + AC $\vdash$ If there are both Moschovakis and strongly compact cardinals, then the first Moschovakis cardinal is smaller than the first strongly compact.

We now ponder particulars.

Kunen has very recently solved the problem of the number of normal measures admitted by a 2VM cardinal.

T 2012 (Scott). ZF + AC $\vdash$ if $\varkappa$ is 2VM, then $\varkappa$ has at least one normal measure.

+ T 2013 (Kunen). ZF + GCH $\vdash$ Let $\varkappa$ be 2VM, $\mu$ a normal measure, $R$ a set of regular infinite cardinals $<\varkappa$ with $\mu(R) = 0$, $F$ an increasing map from $R$ into $\varkappa$ such that $\forall \alpha \in R(F(\alpha)$ is a cardinal $\wedge \mathrm{cf}(F(\alpha)) > \alpha)$, and $\lambda$ a cardinal $\geq \varkappa^{++}$ with $\mathrm{cf}(\lambda) > \varkappa^{+}$. Then there is a Boolean extension of the universe preserving cardinals in which

(i) $2^{\varkappa} = \varkappa^{+}$; $2^{\varkappa^{+}} = \lambda$,

(ii) $\forall \alpha \in R(2^{\alpha} = F(\alpha))$ and

(iii) $\varkappa$ has $\lambda$ distinct normal ultrafilters.

+ P 2014. Let $\varkappa$ be 2VM; can a measure on $\varkappa$ be defined in $V_{\varkappa+n}$ for some $n < \omega$?

T 2015 (Solovay). ZF + AC $\vdash$ If $\varkappa$ is 2VM, there is always at least one normal $\mu$ such that $\mu(\{\alpha < \varkappa \mid \alpha$ is 2VM$) = 0.\}$

Let $\varkappa$ be 2VM, $\mu'$, $\nu'$ normal measures. Denote by $L^{\mu'}$ the universe constructible from $\mu'$, in the sense of Lévy; informally $L^{\mu'}$ is the smallest universe $W$ containing all ordinals and $\mu' \cap W$. Then $\mu = \mu' \cap L^{\mu'}$ is a normal measure on $\varkappa$ in $L^{\mu'} = L^{\mu}$: further, setting $\nu = \nu' \cap L^{\nu'}$,

+ T 2016 (Kunen). $\mu = \nu$;

T 2017 (Silver [*]). GCH is true in $L^{\mu}$, and

+ T 2018 (Kunen). $\mu$ is the only normal measure on $\varkappa$ in $L^{\mu}$.

T 2019. It is also known that $\varkappa$ is the only 2VM cardinal in $L^{\mu}$, and so (by T 2018, which was proved before but follows from T 2016) $\varkappa$ and $\mu$ are definable in $L^{\mu}$.

Formally, T 2017 states that GCH is a theorem of the theory ZF + AC + + $\forall \varkappa, \mu(\mu$ is a normal two-valued $\varkappa$-additive measure on $\varkappa$ and $V = L^{\mu})$; in future this theory will be indicated by ZF + AC + $V = L^{\mu}$.

T 2020 (Solovay). ZF + AC $\vdash$ If $\varkappa$ is supercompact, then there is a normal measure $\mu$ on $\varkappa$ with $\mu(\{\alpha < \varkappa \mid \alpha$ is 2VM$\}) = 1$.

Using this and T 2016, Solovay has proved

$+$ T 2021. ZF $+$ AC $\vdash$ If $\varkappa$ is supercompact, then $\varkappa$ has more than $2^{\varkappa}$ normal measures.

$+$ P 2022. Show that the first strongly compact cardinal $\varkappa$ has a normal measure $\mu$ for which $\mu(\{\alpha < \varkappa \mid \alpha \text{ is 2VM}\}) = 1$.

The first indication that the presence of a 2VM cardinal affected the relationship of $L$ to the universe was

T 2023 (Scott). ZF $+$ AC $+$ MC $\vdash V \neq L$.

(Much more startling things are known now — see T 2033.) A related result has been obtained assuming a strongly compact cardinal. First, a definition:

D 2024. An ultrafilter $U$ on $\lambda$ ($U$ not necessarily $\lambda$ additive) is *uniform* iff

$$\wedge x \subseteq \lambda(\bar{\bar{x}} < \lambda \to x \notin U).$$

$+$ T 2025 (Vopěnka, Hrbáček [H41]). ZF $+$ AC $\vdash$ If there is a uniform ultrafilter on $\lambda^+$ then $V \neq L[a]$ for any $\alpha \subseteq \lambda$.

This result has two corollaries:

T 2026 (Vopěnka, Hrbáček). ZF $+$ AC $+$ SCC $\vdash \wedge y \vee x[x \notin L^y]$.

T 2027 (Vopěnka, Hrbáček). Con(ZF $+$ AC $+$ MC) implies

Con(ZF $+$ AC $+ \vee\varkappa(\varkappa$ is 2VM and there is no uniform ultrafilter on $(2^{\varkappa})^+)$).

$+$ P 2028 (Vopěnka, Hrbáček). Show that ZF $+$ AC $\vdash$ If $\varkappa$ is 2VM, then there is a $\varkappa$-additive uniform ultrafilter on $2^{\varkappa}$.

T 2026 says that if there is a strongly compact cardinal, the universe is not constructible from a set. Let now $\varkappa$ be 2VM, $\mu$ a normal measure on $\varkappa$. By Silver's work (T 2017), $\varkappa$ is 2VM in $L^{\mu}$, and by T 2026, $L^{\mu} \models \neg$ SCC, so that "strongly compact" and "2VM" are not provably the same. That SCC is stronger than MC was shown recently by Kunen, who obtained

T 2029. ZF $+$ AC $+$ SCC $\vdash$ Con(ZF $+$ AC $+$ MC)

as a corollary to a much stronger result, T 2044; he has very recently improved T 2029 to

T 2030. ZF $+$ AC $+$ SCC $\vdash \wedge\alpha \vee M(M$ is a transitive model for ZF $+$ AC $\wedge$ $\wedge\, \alpha \in M \wedge M \models$ "There are $\alpha$ 2VM cardinals").

The following problem, much milder than P 2022, is still open.

$+$ P 2031. Show ZF $+$ AC $+$ SCC $+$ "the first 2VM is not strongly compact" consistent, relative to something else.

One of the more interesting features of large cardinals is their stunning effect on the constructible universe. Gaifman and Rowbottom independently strengthened Scott's theorem T 2023, showing *inter alia* that

T 2032. ZF + AC + MC $\vdash$ there are only countably many constructible reals.

We shall now state the Hauptsatz of Silver's doctoral dissertation, after which we shall note some of the striking consequences and indicate the advances made by Rowbottom and Gaifman.

+ T 2033 (Silver). ZF + AC + $\forall\kappa(\kappa \to (\omega_1)^{<\omega}) \vdash$ For every uncountable cardinal $\alpha$ there is an $X_\alpha \subseteq \alpha$ with the properties

(i) $X_\alpha$ is closed and unbounded in $\alpha$;

(ii) every $x \in L_\alpha$ is definable in $L_\alpha$, allowing parameters form $X_\alpha$, and

(iii) $X_\alpha$ is a set of indiscernibles for $L_\alpha$: viz., if $n \in \omega$; $\kappa_0 < \ldots < \kappa_{n-1}$, $\kappa_0' < \ldots < \kappa_{n-1}'$ are two sequences of elements of $X_\alpha$; and $\mathfrak{A}\ (\mathfrak{x}_0, \ldots, \mathfrak{x}_{n-1})$ is a wff of the languange of set theory with precisely the $n$ free variables shown, then

$$L_\alpha \models \mathfrak{A}[\kappa_0, \ldots, \iota_{n-1}] \leftrightarrow L_\alpha \models \mathfrak{A}[\kappa_0', \ldots, \kappa_{n-1}'].$$

Further, set $X = \bigcup_{\alpha=\alpha>\omega} X_\alpha$; then

(iv) $X$ is closed and unbounded and contains every uncountable cardinal;

(v) $\beta \in X \wedge \beta' \in X \wedge \beta < \beta' . \to L_\beta \prec L_{\beta'}$;

(vi) $\beta \in X \to \beta \cap X$ is a set of indiscernibles for $L_\beta$;

(vii) $\beta \in X \to L_\beta < L$; and

(viii) $X$ is a class of indiscernibles for $L$.

Two striking corollaries;

T 2034 (Rowbottom). ZF + AC + $\forall\kappa(\kappa \to (\omega_1)^{<\omega}) \vdash$ If $\alpha$ is a countable ordinal, then $V_\alpha$ contains only countably many constructible sets.

T 2035. ZF + AC + $\forall\kappa(\kappa \to (\omega_1)^{<\omega}) \vdash$ Let $X$ be as in T 2033. Then the first $\alpha \in X$ is countable; and every $\alpha \in X$ is weakly compact and $\to (\omega)^{<\omega}$ in $L$.

Gaifman working in the theory ZF + AC + MC had noticed that if $\alpha, \beta$ are uncountable cardinals and $\alpha < \beta$, then $L_\alpha \prec L_\beta \prec L$. Rowbottom had realized the significance of the Ramsey properties, had strenthened Erdős and Hajnal's theorem that $\kappa$ 2VM $\to \kappa$ Ramsey to

+ T 2036. ZF + AC $\vdash$ Let $\kappa$ be 2VM, $\mu$ a normal measure. Let $f\colon [\kappa]^{<\omega} \to \lambda$, where $\lambda < \kappa$, be a partition. Then there is a $y \subseteq \kappa$ with $\mu(y) = 1$ which is a set of indiscernibles for $f$

and had noted that if $\bigvee \varkappa(\varkappa \neq (\omega_1)^{<\omega})$, then there is a non-constructible real. Moreover, he formulated the notion of a *remarkable* set of sentences, one of the key concepts in the proof of T 2033 (cf. Silver's thesis). A model-theoretic property of measurable cardinals established by Rowbottom is discussed in Section 4.

D 2037 (Solovay [12]). Assume $\bigvee \varkappa(\varkappa \to (\omega_1)^{<\omega})$. Define

$$O^{\#} = \text{the theory of } \langle L_{\aleph_\omega}, \in, \aleph_1, \aleph_2, \ldots, \aleph_n, \ldots \rangle_{n<\omega},$$

(where $\aleph_1 =$ the real $\aleph_i$).

By T 2033, $L_{\aleph_\omega} \prec L$, an the $\aleph_n$ $(1 \leq n < \omega)$ are a set of indiscernibles for $L$: so that $O^{\#}$. which by suitably Gödel-numbering the language may be regarded as a set of integers, is "the theory of $L$ with $\omega$ indiscernibles".

T 2038 (Solovay). There is a $\pi^1_2$-predicate $A$ of reals such that $\mathrm{ZF} \vdash \bigvee x A(sx) \to \to \bigvee! x A(x)$, and such that $\mathrm{ZF} + \mathrm{AC} + \bigvee \varkappa(\lambda \to (\omega^{<\omega})) \vdash A(O^{\#})$. Furthermore, the assertion that $\bigvee x A(x)$ is equivalent in ZF to the conclusions of T 2033.

Instead of $\bigvee x A(x)$, we shall say "$O^{\#}$ exists". Thus if $O^{\#}$ exists, there is a class $X$ of ordinals satisfying (i)—(viii) of T 2033, where $X_\alpha = X \cap \alpha$ for an uncountable cardinal $\alpha$. It follows from T 2038 that $O^{\#}$, if it exists, is a non-constructible real which is $\Delta^1_3$, and in which every constructible real is recursive. As the theory of $L$ may be read off from $O^{\#}$, it may be shown that every ordinal definable in $L$ is either finite or isomorphic to a $\Delta^1_3$ well ordering of $\omega$, and so countable, In the theory $\mathrm{ZF} + O^{\#}$ exists", one may prove Rowbottom's theorem T 2034 and that every $\alpha \in X$ is strongly Mahlo in $L$.

The moral of the next theorem is that two measurable cardinals are much better than one.

T 2039 (Solovay). $\mathrm{ZF} + \mathrm{AC} \vdash$ Suppose $\varkappa$ and $\lambda$ are two measurable cardinals with $\varkappa < \lambda$, and let $\mu$ be a normal measure on $\varkappa$. Then there is an $X \subseteq \varkappa$ and a $Y \subseteq On$ such that

(i) $X$ is closed and unbounded in $\varkappa$,

(ii) $Y$ is closed and unbounded, and $\varkappa < \min(Y)$,

(iii) $X \cup Y$ contains every uncountable cardinal except, $\varkappa$,

(iv) if $\mathfrak{A}(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is any formula of the language of set theory (with predicates $\in$, $\equiv$) enriched by a name for $\varkappa$ and a name for $\mu$, with precisely the free variables shown, and if $\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_n$, are in $X$, with $\alpha_1 < \ldots < \alpha_n$, $\alpha'_1 < \ldots < \alpha'_n$, and $\beta_1, \ldots, \beta_m, \beta_1, \ldots, \beta'_m$ are in $Y$, with $\beta_1 < \ldots < \beta_n$, $\beta'_1 < \ldots < \beta'_m$, then

$$L \vdash \mathfrak{A}[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m] \leftrightarrow L^{\mu} \vdash \mathfrak{A}[\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_m],$$

and

(v) every element of $L^\mu$ is definable in $L^\mu$, allowing parameteres from $X \cup Y$.

Properties akin to the more precise clauses of T 2033 may be established for $L_\alpha$, where $\alpha$ is an uncountable cardinal. In particular, we may use the analogue for (vii) to formalize the following in set theory:

D 2040. $O^\dagger$ = the theory of $(L^\mu, \in, \varkappa, \mu, \aleph_1, \aleph_2, \ldots, \varkappa^+, \varkappa^{++}, \ldots)$. $O^\dagger$ is the theory of $L^\mu$ with $\omega$ indiscernibles from $X$ and $\omega$ indiscernibles from $Y$.

T 2041 (Solovay). There is a $\pi^1_2$ predicate of reals $B$ such that $\mathrm{ZF} \vdash \bigvee x B(x) \rightarrow$ $\rightarrow \bigvee\, | x B(x)$ and such that $\mathrm{ZF} + \mathrm{AC} \vdash$ If $\varkappa$ is 2VM and $\varkappa < \lambda$ and $\lambda$ is 2VM, then $B(O^\dagger)$ where $O^\dagger$ is defined by D 2040. We shall write $\bigvee x B(x)$ as "$O^\dagger$ exists".

D2042. $\mathfrak{F}_\alpha$ = the filter generated by the closed unbounded subsets of $\alpha$; $U_\alpha = \mathfrak{F}_\alpha \cap L^{\mathfrak{F}_\alpha}$.

T 2043 (Solovay). $\mathrm{ZF} \vdash$ Suppose $O^\dagger$ exists. Then for every uncountable aleph $\alpha$, $U_\alpha$ is a normal ultrafilter on $\alpha$ in $L^{\mathfrak{F}_\alpha}$: so that $\alpha$ is 2VM in $L^{\mathfrak{F}_\alpha} = L^{U_\alpha}$. Further $\alpha^+$ is strongly inaccessible in $L^{\mathfrak{F}_\alpha}$, so that the existence of a standard model of ZF + AC + MC may be proved.

T 2044 (Kunen). ZF + AC + SCC $\vdash$ $O^\dagger$ exists.

T 2029 is an immediate consequence fo T 2043 and T 2044.

Chang has investigated large cardinal axioms using infinitary languages. $L$ may be defined hierarchically using the ordinary language of set theory, $L_{\omega,\omega}$. Let $M^\alpha_{\varkappa,\varkappa}$ denote the $\alpha$-th level of the corresponding hiearchy for $L_{\varkappa,\varkappa}$. The $L_\alpha$ mentioned above is in this new notation $M^\alpha_{\omega,\omega}$. Write $M^{\mathrm{On}}_{\varkappa,\varkappa} = \bigcup\limits_{\alpha \in \mathrm{On}} M^\alpha_{\varkappa,\varkappa}$. Then

T 2045 (Solovay; Lévy; Silver). ZF + AC $\vdash$ $\varkappa$2VM $\rightarrow$ $M^{\mathrm{On}}_{\varkappa^+,\varkappa^+} \neq V$.

T 2046 (Chang). ZF + AC + $\bigvee \varkappa(\varkappa \rightarrow (\omega_1)^{<\omega}) \vdash$ Let $\lambda$, $\mu$, $\nu$ be infinite regular cardinals with $\lambda < \mu < \nu$. Then $M_{\omega,\omega} \prec M^\nu_{\omega,\omega}$.

+ P 2047 (Chang). Show

$$\mathrm{ZF} + \mathrm{AC} + \bigvee \varkappa(\varkappa(\varkappa))\ V \neq M^{\mathrm{On}}_{\omega_1,\omega_1}.$$

T 2047 (Silver). Con(ZF + AC + MC) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \mathrm{MC} + \neg\, M^{\omega_2}_{\omega_1,\omega} \prec M^{\mathrm{On}}_{\omega_1,\omega_1}).$$

Further results and problems are given in CHANG's lecture notes [*]. We now discuss real-valued measurable cardinals. Inspired by the work of HANF [3] on weakly compact cardinals, KEISLER and TARSKI [4] proved

T 2048. ZF + AC $\vdash$ If $\varkappa$ is 2VM, then $\varkappa$ is strongly Mahlo.

Three people independently adapted the methods of Keisler and Tarski to show that RVM's enjoy a similar remoteness:

T 2049 (SOLOVAY; JENSEN; FREMLIN). ZF + AC$\vdash$ Let $\varkappa$ be RVM. Then $\varkappa$ has a normal measure, and is weakly Mahlo.

Solovay, indeed, obtained these results in a more general setting.

+ T 2050 (SOLOVAY). ZF + AC$\vdash$ Suppose $i(\varkappa, \varkappa)$. Then $\varkappa$ has a normal $\varkappa$-complete $\varkappa$-saturated ideal $\mathfrak{Z}$, and for any $B \subseteq \varkappa$, $B \notin \mathfrak{Z}$,

$$\{\alpha < \varkappa | \alpha \cdot \text{ is not } B\text{-Mahlo}\} \in \mathfrak{Z}.$$

Further, $V \neq L$. Moreover, if $i(\varkappa, \lambda)$ for some $\lambda < \varkappa$, then $\varkappa$ has a normal $\varkappa$-complete $\lambda$-saturated ideal, $\mathfrak{Z}$, say. $L^{\mathfrak{Z}}$ satisfies GCH above $\lambda$, and $\mathfrak{Z} \cap L^{\mathfrak{Z}}$ is a $\lambda$-saturated ideal in $L^{\mathfrak{Z}}$: it follows that $\varkappa$ is 2VM in $L^{\mathfrak{Z}}$.

Indeed, Kunen's result T 2016 shows that $\mathfrak{Z} \cap L^{\mathfrak{Z}}$ is a normal prime ideal, and so the full GCH is true in $L^{\mathfrak{Z}}$.

P 2051. What are the possible cardinalities of minimal generating sets for the measure algebra of a real-valued measurable cardinal, or for an $\omega_1$-saturated Boolean-valued measurable cardinal?

If $\varkappa = 2^{\aleph_0}$ is RVM, then every minimal generating set for its measure algebra is of cardinality $> \varkappa$.

+ D 2052. $k(\varkappa)$ iff $i(\varkappa, \varkappa) \ .\wedge. \ \forall \lambda < \varkappa(\neg i(\varkappa, \lambda))$.

T 2053 (LÉVY). ZF + AC $\vdash$ $\varkappa$ strongly inaccessible and $k(\varkappa).\rightarrow$ $\varkappa$ not weakly compact.

It is readily seen that $k(\varkappa) \rightarrow \varkappa > \omega$. For the next two questions, consistency results and proofs from large cardinals are both interesting.

P 2054. Is there a $\varkappa$ such that $k(\varkappa)$?

+ P 2055. Is there a strongly inaccessible $\varkappa$ such that $k(\varkappa)$?

P 2056. Does ZF + GHC $\vdash \neg \forall \varkappa k(\varkappa)$?

+ P 2057. Prove Con(ZF + AC + $\omega_1$ has an $\omega_1$-complete ideal $I$ (even better: $\omega_2$-saturated) with $S(\omega_1)/I$ complete), assuming Con(ZD + AC + some large cardinal).

Part of T 2050 was proved in Solovay's second lecture [*]; in his last, he applied properties of saturated ideals to solve a problem of Fodor and Hajnal:

D 2058. Let $\varkappa$ a cardinal. $B \subseteq \varkappa$ is *stationary* if it intersects every closed and unbounded subset of $\varkappa$.

+ T 2059 (SOLOVAY [*]). ZF + AC $\vdash$ Let $\varkappa$ be an uncountable cardinal. Then every stationary subset of $\varkappa$ is the union of $\varkappa$ disjoint stationary subsets of $\varkappa$.

P 2060 (ERDŐS). Is there a stationary subset of $\aleph_1$ which is the union of $2^{\aleph_1}$ stationary subsets the intersection of any two of which is not stationary?

Lastly, a problem about a simplified form of the Moschovakis property.
+ P 2061. Does $\omega_{\omega 2} \to (2)^1_2$?

## § 1. *Relative Consistency Results*

T 2100 (SILVER). Con(ZF + AC + $\bigvee \varkappa(\varkappa \to (\omega)^{<\omega})$ implies

$$\mathrm{Con}(\mathrm{ZF} + V = L + \bigvee \varkappa(\varkappa \to (\omega)^{<\omega}).$$

This undoes one weakening of the hypothesis of T 2033 and is a corollary of

T 2101 (SILVER). ZF + AC $\vdash$ If $\varkappa \to (\omega^{<\omega})$ the it is true in $L$ that $\varkappa \to (\omega)^{<\omega}$.

T 2102. TD + AC $\vdash$ If $\varkappa$ is weakly compact, then in $L$ it is true that $\varkappa$ is weakly compact.

T 2103. Con(ZF + AC + there is a weakly compact cardinal) implies Con(ZF + + $V = L$ + there is a weakly compact cardinal).

The problem of the relative consistency of CH to MC has been settled by LÉVY and SOLOVAY [7]: the results

T 2104. Con(ZF + AC + MC) implies Con(ZF + AC + MC + CH) and

T 2105. Con(ZF + AC + MC) implies Con(ZF + AC + MC + $\neg$ CH).

follow from their basic lemma that measurable cardinals remain measurable under small extensions by forcing:

T 2106 (LÉVY; SOLOVAY). ZF + AC $\vdash$ Let $\varkappa$ be 2VM and **B** a complete Boolean algebra of cardinality $< \varkappa$. Then

$$[\check{\varkappa} \text{ is 2VM}]^{\mathbf{B}} = \mathbf{1}.$$

Thus for example both SH and its negation are consistent relative to ZF + AC + MC. The following theorems give similar results for the appropriate theories:

T 2107 (McALOON). ZF + AC $\vdash$ Let $\varkappa$ be strongly compact, and $\mathbf{B}$ a complete Boolean algebra of cardinality $<\varkappa$; then

$$[\check{\varkappa} \text{ is strongly compact}]^{\mathbf{B}} = \mathbf{1}.$$

T 2108. ZF + AC $\vdash$ Let $\alpha$ be a limit ordinal for which $\bigvee\lambda(\lambda \to (\alpha)^{<\omega})$, and $\mathbf{B}$ a complete Boolean algebra of cardinality $<\varkappa = \varkappa(\alpha)$. Then

$$[\varkappa \to (\alpha)^{<\omega}]^{\mathbf{B}} = \mathbf{1}.$$

In particular, if $\alpha$ is a Ramsey cardinal, then $\varkappa \to \alpha$ and

$$[\alpha \text{ is a Ramsey cardinal}]^{\mathbf{B}} = \mathbf{1}.$$

T 2108 is proved by applying T 2008 with $\nu = \overline{\overline{\mathbf{B}}}$

The theorem

+ T 2109 (JENSEN; SILVER). Con(ZF + AC + MC) implies Con(ZF + GCH + + MC)

is a consequence of both T 2017 and T 2110. Silver's method is to contract the universe; Jensen expands it.

T 2110 (JENSEN). ZF $\vdash$ Every c.s.m. $M$ of ZF + AC can be extended to a c.s.m. $N$ of ZF + GCH which has the further properties

(i) $O_n^N = O_n^M$;

(ii) if $\varkappa$ is 2VM in $M$, then $\varkappa$ is 2VM in $N$;

(iii) if $\lambda$ is a limit ordinal, $\varkappa$ an ordinal and $\varkappa \to (\lambda)^{<\omega}$ in $M$, then $\varkappa \to (\lambda)^{<\omega}$ in $N$;

in particular,

(iv) $\varkappa$ Ramsey in $M$ implies $\varkappa$ Ramsey in $N$;

and

(v) if $\varkappa \to (\omega_1)^{<\omega}$ in $M$, then $\varkappa \to (\omega_1)^{<\omega}$ in $N$.

Clause (v) holds because $\omega_1$ is preserved in the extension. Standard techniques translate the proof of T 2100 into proofs of

T 2111 (JENSEN). Con(ZF + AC + there is a Ramsey cardinal) implies

$$\text{Con(ZF + GCH + there is a Ramsey cardinal).}$$

T 2112 (JENSEN). Con(ZF + AC + $\bigvee\varkappa(\varkappa \to (\omega_1)^{<\omega})$) implies

$$\text{Con(ZF + GCH + } \bigvee\varkappa(\varkappa \to (\omega_1)^{<\omega})).$$

P 2113. Does Con(ZF + AC + SCC) imply Con(ZF + GCH + SCC)?

+ P 2114. Does Con(ZF + AC + SiCC) imply Con(ZF + GCH + Succ)?

+ T 2115 (Scott). $\mathrm{ZF} + \mathrm{AC} \vdash \kappa\ \mathrm{2VM} \wedge \bigvee \lambda < \kappa 2^\lambda = \lambda^+ . \rightarrow 2^\kappa = \kappa^+$. More generally, if $\kappa$ is 2VM, $\mu$ a normal measure and $\mu\{\lambda < \kappa \mid \lambda = \lambda \wedge 2^\lambda = \lambda^+\} = 1$ then $2^\kappa = \kappa^+$.

A number of results of the following type may also be proved.

T 2116 (Vopěnka [H27]). $\mathrm{ZF} + \mathrm{AC} \vdash$ Let $\kappa$ be 2VM. Then

(i) if $\gamma < \kappa$ and $\bigvee \alpha \leq \kappa 2^{\aleph_\alpha} \leq \aleph_{\alpha+\gamma}$, then $2^\kappa \leq \aleph_{\kappa+\gamma}$;
(ii) if $\bigvee \alpha < \kappa\, 2^{\aleph_\alpha} \leq \aleph_{\alpha+\alpha}$, then $2^\kappa \leq \aleph_{\kappa+\kappa}$.

T 2115 and T 2116 relate to the following important questions.

+ P 2117. Does $\mathrm{ZF} + \mathrm{AC} \vdash \kappa\ \mathrm{2VM} \rightarrow 2^\kappa = \kappa^+$?

+ T 2118 (Příkrý). $\mathrm{ZF} \vdash$ Let $M$ be a c.s.m. of ZF + AC + MC, and let $\kappa$ be 2VM in $M$; then there is a c.s.n. $N$ of ZF + AC extending $M$, sich that

(i) $O_n^N = O_m^M$;
(ii) $\mathrm{Card}^N = \mathrm{Card}^M$;
(iii) for all cardinals $\lambda < \kappa$, $S^N(\lambda) = S^M(\lambda)$; and
(iv) $\mathrm{cf}^N(\kappa) = \omega$.

To prove this we use forcing to add an ascending $\omega$-sequence of ordinals, $f$, with $f''\omega$ cofinal in $\kappa$. A condition is a pair, the first element of which is a finite ascending sequence of ordinals $\alpha_0 < \ldots < \alpha_n < \kappa$; the second is a set $A \subseteq \kappa$, with $\mu(A) = 1$, where $\mu$ is some fixed normal measure on $\kappa$, and $\lambda \in A \rightarrow \alpha_n < \lambda$. The intended interpretation is "$f(0) = \alpha_0 \wedge \ldots \wedge f(n) = \alpha_n \wedge \bigvee m > nF(m) \in A$", and the set of conditions is partially ordered accordingly. Evidently in the extension $f$ is an $\omega$-sequence cofinal in $\kappa$; that no new subsets of earlier ordinals are introduced follows from the lemma:

T 2119 (Příkrý). Let $\mathfrak{A}$ be a sentence of the forcing language, and $\langle\langle \alpha_0, \ldots, \alpha_n\rangle, A\rangle$ a condition; then there is an $A' \subseteq A$ of $\mu$-measure 1 such that $\langle\langle \alpha_0, \ldots, \alpha_n\rangle, A'\rangle \Vdash \mathfrak{A}$ or $\langle\langle \alpha_0, \ldots, \alpha_\alpha\rangle, A'\rangle \Vdash \neg\, \mathfrak{A}$,

which is proved using the Ramsey properties of $\kappa$.

T 2118 gives a partial solution to P 1207 and, if the answer to P 2117 is negative, may lead to a solution of the singular cardinals problem. Příkrý has proved the following by related methods:

T 2120. $\mathrm{ZF} \vdash$ Let $M$ be a c.s.m. of ZF + AC + SCC. Let $\kappa$ be strongly compact in $M$, and let $\lambda \in M$ be a regular cardinal greater than $\kappa$. Then there is

an extension $N$ of $M$ with the same ordinals such that

(i) $N$ is a model of ZF + AC;
(ii) $\bigvee \alpha < \varkappa (S^N(\alpha) = S^M(\alpha))$;
(iii) $\lambda^N = \varkappa$;
(iv) $\mathrm{cf}^N(\varkappa) = \omega$; and
(v) $\bigwedge \beta > \lambda$ ($\beta$ a cardinal in $M \to \beta$ a cardinal in $N$).

Thus a negative answer to the following would be useful:

+ P 2121. Does ZF + AC $\vdash \varkappa$ strongly compact $\to \bigwedge \lambda < \varkappa (2^\lambda = \lambda^+)$?

P 2122. Does ZF + AC $\vdash \varkappa$ weakly compact $\to 2^\varkappa = \varkappa^+$?

+ T 2123 (KUNEN). Let $\varkappa$ be 2VM; then $\{\alpha < \varkappa \mid 2^{\aleph_\alpha} > \aleph_{\alpha+1}\}$ may be of normal measure 1.

For by T 2013 (iii) $\varkappa$ may have two normal measures $\mu$ and $\nu$; apply T 2013 (ii) with an $R$ for which $j(R) = 0$, $\nu(R) = 1$.

By T 2049 and T 2003 we know that GCH implies that there are no RVM cardinals: but

+ T 2124 (SOLOVAY). Con(ZF + AC + MC) implies and is implied by Con(ZF + + AC + $2^{\aleph_0}$ is RCVM).

We give a sketch of the argument. Silver's proof of T 2017 is the paradigm for one direction: if $\mu$ is a normal real-valued measure on $\lambda$, form $L^\mu$. The theorem of Rowbottom used in Silver's proof, T 4009, can be adapted to show that GCH holds in $L^\mu$; or we can use Kunen's remark following T 2050 that $\mu \cap L^\mu$ is a two-valued measure in $L^\mu$. The method for the other direction parallels an earlier proof of Příkry where $\varkappa$ generic reals were added to a model in which $\varkappa$ was 2VM: in the extension $2^{\aleph_0} = \varkappa$, $i(\varkappa, \omega_1)$, but $\varkappa$ was not RVM. Thus

T 2125 (PŘÍKRY). Con(ZF + AC + MC) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + W\mathrm{RMC} + i(2^{\aleph_0}, \aleph_1)).$$

T 2124 is proved by adding $\varkappa$ random reals: then in the extension $2^{\aleph_0} = \varkappa$ and $\varkappa$ is RVM.

T 2126 (MCALOON). Con(ZF + AC + MC) implies Con(ZF + GCH + MC + + $V = K$).

T 2127 (MCALOON). Con(ZF + AC + MC) implies Con(ZF + GCH + MC + + $V \neq K$). ($K$ is the universe of hereditarily ordinal definable sets).

P 2128. Does Con(ZF + AC + SCC) imply Con(ZF + $V = K$ + SCC)?

## Section 3. The structure of the real line

If ZF is consistent, then it is impossible to prove that every real is constructible. It is natural to seek the "level of complexity" at which non-constructible reals may first occur. Similarly, one may ask "What is the simplest Lebesgue non-measurable set of reals?". Partial answers to these and related questions have been obtained, when "level of complexity" is taken to mean level of the analytical or projective hierarchy; some are deductions from large cardinal axioms, like "If there is a measurable cardinal, then every $\mathbf{\Sigma}^2_1$ set of reals is Lebesgue measurable"; some are relative consistency results. These results are arranged in five paragraphs: in § 0 the projective hierarchy is discussed, and recent applications of the axiom of determinacy reported; in § 1, answers are given to the question "What reals are constructible". § 2. expounds the work of Mansfield and Solovay on perfect subsets of sets of reals. § 3 deals with Lebesgue measure and the property of Baire, and in § 4, some problems raised by Hausdorff are treated.

### § 0. *The Projective Hierarchy*

For notation see ADDISON [*].

An important problem is the pattern of the separation principles for the projective hierarchy. We write Sep($\mathcal{C}$), (Red($\mathcal{C}$)) for "$\mathcal{C}$ has the first separation (the reduction) property"; Unif($\mathcal{C}$) for "every set in $\mathcal{C}$ can be uniformized by a set in $\mathcal{C}$." It is easily seen that in ZF, for every $n < \omega$, Unif($\mathbf{\Pi}^1_n$) implies Unif($\mathbf{\Sigma}^1_{n+1}$), that for any class $\mathcal{C}$, Red($\mathcal{C}$) implies Sep($\mathcal{C}$)$^-$, and that for all reasonable $\mathcal{C}$ (in particular the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$), Unif($\mathcal{C}$) implies Red($\mathcal{C}$).

T 3000 (KONDO–ADDISON). ZF $\vdash$ Unif($\mathbf{\Pi}^1_1$): hence

T 3001 ZF $\vdash$ Sep ($\mathbf{\Sigma}^1_1$) $\wedge$ Sep($\mathbf{\Pi}^1_2$).

T 3002 (ADDISON). ZF + $V = L \vdash \forall n \geq 2$ Unif($\mathbf{\Sigma}^1_n$).

T 3003 (GÖDEL). ZF + $V = L \vdash$ There is a $\Delta^1_2$ well ordering of the reals of order type $\omega_1$.

The well ordering $\prec$ constructed by Gödel to prove T 3003 has the useful property that the quantifier $\wedge y \preceq x$ (for any $x \subseteq \omega$) when applied to a $\Delta^1_2$ formula yields a $\Delta^1_2$ formula: this fact is used to prove T 3002, of which an immediate corollary is

T 3004 (ADDISON). ZF + $V = L \vdash \wedge n \geq 2$ Sep ($\mathbf{\Pi}^1_n$).

By AD we mean the axiom of determinacy, of Mycielski and Steinhaus, discussed in Solovay's lecture [*]. Addison and Moschovakis and, indepen-

dently, Martin have recently shown

T 3005. $\mathrm{ZF} + \mathrm{AD} \vdash \bigwedge n(\mathrm{Red}(\mathbf{\Sigma}^1_{2n}) \wedge \mathrm{Red}(\mathbf{\Pi}^1_{2+n}))$,

and so

T 3006. $\mathrm{ZF} + \mathrm{AD} \vdash \bigwedge n(\mathrm{Sep}(\mathbf{\Pi}^1_{2n}) \wedge \mathrm{Sep}(\mathbf{\Sigma}^1_{2+n}))$.

Let us pause to note some other consequences of AD.

T 3007 (Mycielski, Swierczkowski). $\mathrm{ZF} + \mathrm{AD} \vdash$ All sets of reals are Lebesgue measurable.

D 3008. A set of reals has the *property of Baire* iff it is equal to an open set up to first category.

Now every Borel set has the property of Baire: hence

T 3009. A set of reals has the property of Baire iff it is equal to a Borel set up to first category.

T 3010. A set of reals is Lebegue measurable iff it is equal to a Borel sep up to a set of measure 0.

Henceforth we shall write "LM" for "Lebesgue measurable" and "is PB" for "has the property of Baire." The duality suggested by T 3009, 3010 will be brought out in § 3.

T 3011 (Mycielski). $\mathrm{ZF} + \mathrm{AD} \vdash$ Every uncountable subset of the reals is PB and has a perfect subset; hence $\aleph_0$ adj $2^{\aleph_0}$.

T 3012 (Solovay [*]). $\mathrm{ZF} + \mathrm{AD}W$ Every subset of $\aleph_1$ is constructible from a real, whence every subset of $\aleph_1$ contains or is disjoint from a closed unbounded subset,

and so

T 3013 (Solovay). $\mathrm{ZF} + \mathrm{AD} \vdash \aleph_1$ is 2VM.

A simpler proof by Martin of T 3013 is based on

T 3014 (Martin). $\mathrm{ZF} + \mathrm{AD} \vdash$ Let $\mathcal{E} \subseteq \mathcal{D}$ the set of all degrees of insolubility. There is a degree $d_0$ such that either $\bigwedge d \geq d_0(d \in \mathcal{E})$ or $\bigwedge d \geq d_0(d \notin \mathcal{E})$.

$\mathcal{E}$ defines a game: I wins if the play has degree in $\mathcal{E}$, II otherwise. Let $d_0$ be the degree of a winning strategy: if I wins, then $\bigwedge d \geq d_0(d \in \mathcal{E})$; otherwise $\bigwedge d \geq d_0(d \notin \mathcal{E})$. Let $\mu(\mathcal{E}) = 1$ in the first case, $= 0$ in the second. Then $\mu$ is a measure on $\mathcal{D}$: it is countably additive because no $\omega$-sequence of degrees can be cofinal in $\mathcal{D}$. T 3014 also leads to a reasonably short proof of

+ T 3015 (Solovay). $\mathrm{ZF} + \mathrm{AD} \vdash \aleph_2$ is 2 VM;

as a corollary,

T 3016. ZF + AD $\vdash$ Con(ZF + AC + MC).

Using T 3015 and the methods of T 2041, we can show that the class of cardinals $\varkappa > \aleph_1$ forms a class of indiscernibles for $L^{\mu'}$, where $\mu'$ is the canonical measure on $\aleph_1$ obtained from the $\mu$ above.

D 3017. We write ADD for the assertion that every set of reals ordinal definable from a real is determinate.

The interest in ADD is that T 3005 and T 3006 may be proved in ZF + + ADD, and that ADD is not known to violate AC. Indeed

T 3018 (SOLOVAY). Con(ZF + AD + $DC^{\omega}$) implies and is implied by

Con(ZF + ADD + AC).

T 3019 (Con(ZF + AD) implies and is implied by Con(ZF + ADD).

P 3020 Prove that Con(ZF + AD) implies Con(ZF + AD + $DC^{\omega}$).

It is known that all open games, and more generally, all $F_{\sigma\delta}$ games are determined.

+ P 3021. Does ZF + DCo $\vdash$ Every Borel game is determined?

Martin has recently made important progress by showing that

+ T 3022. ZF + AC + MC $\vdash$ Every $\mathbf{\Sigma}^1_1$ game is determined.

In fact MC can be weakened to $\forall\varkappa(\varkappa \to (\omega_1)^{<\omega})$; and

T 2023. ZF + AC $\vdash$ If $\forall\varkappa\wedge\alpha < \omega_1(\varkappa \to (\alpha)^{<\omega})$, then every Borel game is determined.

It is worth remarking here that the statement "every Borel game is determined" relativizes to $L$ by Shoenfield's theorem T 3100; and "$\forall\varkappa(\varkappa \to (\alpha)^{<\omega}$ for every countable $\alpha$)" is true in $L$ if true in $V$ by Silver's result that if $\alpha$ is constructibly countable and $\varkappa \to (\alpha)^{<\omega}$, then in $L$, $\varkappa \to (\alpha)^{<\omega}$.

Friedman has shown that set theoretical arguments are necessary to prove that every Borel game is determinate. Let us call second order arithmetic with the full comprehension schema *analysis*. The assertion that every Borel game has a strategy may be formulated as a schema in analysis. Then Friedman's theorem states that

+ T 3024. If analysis is consistent, then so is analysis + "there is a $\Delta^0_3$ indeterminate Borel set."

+ T 3025 (MOSCHOVAKIS; PŘÍKRY). ZF + AC $\vdash$ If there is a Moschovakis cardinal, then every $\mathbf{\Sigma}^1_2$ game is determined.

It is easy to see, by considering $L^{\mu}$, where $\mu$ is a normal measure on $\varkappa$, that Con(ZF + AC + MC) implies Con(ZF + AC + MC + not every $\mathbf{\Sigma}^1_2$ game is determined).

+ P 3026 (ADDISON). Show ZF + AD $\vdash \bigvee n\ \mathrm{Unif}(\mathbf{\Pi}^1_{2n+1})$.

T 3027. ZF + $V = L^{\mu}$ Unif($\mathbf{\Sigma}^1_3$).

Red($\mathbf{\Sigma}^1_3$) and Sep($\mathbf{\Sigma}^1_3$) cannot hold together; so T 3027 shows that Sep($\mathbf{\Sigma}^1_3$) cannot be proved in ZF + AC + MC.

P 3028. Show ZF + AC + some large cardinal $\vdash$ Unif($\mathbf{\Pi}^1_3$).

P 3029. Show Con(ZF + AC + Sep($\mathbf{\Sigma}^1_3$)), assuming anything.

+ P 3030. Show Con(ZF + AC + $\neg$ Sep($\mathbf{\Sigma}$) + $\neg$ Sep($\mathbf{\Pi}^1_3$)), assuming anything.

## § 1. *Constructibility of Reals*

T 3100 (MOSTOWSKI). ZF $\vdash$ If $M$ is a transitive model of ZF, then any $\Sigma^1_1$ statement, allowing real parameters from $N$, is true in $M$ iff it is true in the universe.

T 3101 (SHOENFIELD). ZF $\vdash$ If $M$ is a transitive model containing all countable ordinals, then any $\Sigma^1_1$ statement, with real parameters from $M$, is true in $M$ iff it holds in the universe.

T 3100, 3101 holds also when $M$ is a class term defining an inner model containing all ordinals. In particular, taking $M = L$, in T 3102, and $M = L[b]$ in T 3103, we have

T 3102 (SHOENFIELD). ZF $\vdash$ Every $\Sigma^1_2$ or $\Pi^1_2$ real is constructible;

T 3103 (SHOENFIELD). ZF $\vdash a \subseteq \omega \wedge b \subseteq \omega \wedge a \Sigma^1_2$ in $b. \rightarrow a \leq_L b$.

+ Moschovakis has remarked that if $M \subseteq N$ are transitive models of ZF with the same ordinals, and if both $M$ and $N$ satisfy "Every $\mathbf{\Delta}^1_2$ game is determined" then every $\Sigma^1_3$ or $\Pi^1_3$ sentence with parameters from $M$ is true in $M$ iff it is true in $N$; and hat raised the general problem of finding conditions on $N$ that would ensure that $\Sigma^1_n$ and $\Pi^1_n$ sentence relativize from $N$ to $M$.

In Section 2 we remarked Solovay's theorem that ZF + AC + MC $\vdash$ there is a $\Delta^1_3$ non-constructible real; Solovay has recently shown by a relative consistency result that Shoenfield's theorem T 3102 is best possible. The sequence of events is this:

T 3104 (KRIPKE, MARTIN, SACKS). Con(ZF) implies Con(ZF + $\neg$ AC + there is a non-constructible $\Delta^1_4$ real).

+ T 3105 (SOLOVAY). Con(ZF) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \forall x \subseteq \omega(x \notin L \wedge V = L[x] \wedge x \in \Delta^1_3 \wedge \omega_1^L = \omega_1 \wedge$$

$$x \text{ is the unique solution of a } \Pi^1_2 \text{ predicate})).$$

T 3106 (JENSEN). Con(ZF) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \forall x \subseteq \omega(x \notin L \wedge V = L[x] \wedge x \in \Delta^1_4 \wedge x \text{ is the}$$

$$\text{unique solution of a } \Pi^1_3 \text{ predicate} \wedge \omega_1^L = \omega_1)).$$

T 3107 (JENSEN). Con(ZF) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \forall x \subseteq \omega(x \notin L \wedge V = L[x] \wedge x \in \Delta^1_4 \wedge x \text{ is the unique}$$

$$\text{solution of a } \Pi^1_3 \text{ predicate}) \wedge \bigwedge y \subseteq \omega(y \in L \to y \in \Delta^1_4)).$$

Jensen obtained T 3106 in ignorance of Solovay's result 3105, and about a week later. Jensen's method for 3106 is to construct a model in which the

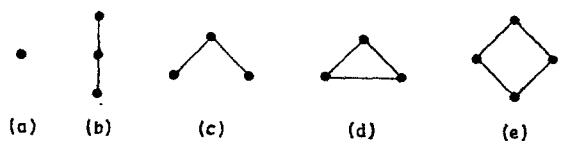



*Fig. 2*

$L$-degrees form a pattern with $\omega + 1$ levels; each level has either a single point (Fig. 2/a) or a triangle (Fig. 2/d), thus the situation seen on Fig. 3 is yielded.

The model contains a real $t$, of the maximum $L$-degree, with $t = \{n$: the $n$-th level of type Fig. 2/c$\}$. This is done by iteration of Sacks forcing. (Martin showed that models with $L$-degree patterns on Fig. 2/b and 2/e could be constructed by appropriate iterations.)

3107 is proved similarly, but the pattern is drawn in the degrees of *collapsing functions*, viz. functions mapping $\omega$ onto $\omega_1^L$. The notion used here, that of minimal collapsing function, is due to Příkry.

3105 is proved by a different method. First Solovay shows that $\omega$ predicates $p_i$ of reals can be defined such that for each $i < \omega$, $\mathrm{ZF} + V = L \vdash \neg\, \forall x \subseteq \omega P_i(x)$. Now let $M$ be a c.s.m. of $\mathrm{ZF} + V = L$: then a sequence $\langle b_i \mid i < \omega\rangle$ of reals is added to $M$ to form a model $N$ of ZF + AC, and such that for each $i$, $P_i(b_i)$ holds in $N$; but if $N_i = M[\langle b_j \mid j \neq i\rangle]$, then $N_i \models \neg\, \forall x \subseteq \omega P_i(x)$. Thus the predicates $P_i$ give $\omega$ "independent" notions of genericity over $L$. Now following McAloon's methods for obtaining models of $V = K$, a set $t \subseteq \omega$ is constructed such that $M[\langle b_i \mid i \in t\rangle] =_{df} M'$ is the desired model:

$t$ codes the construction of $M[\langle b_i \mid i \in t\rangle]$ from $M$, and in $M'$, $t$ will be a non-constructible $\Delta^1_3$ real; $M' = M[t]$.

$+$ P 3108. Find a model in which $V = L[x]$, some $x \subseteq \omega$, $x \notin L$, $x$ of minimal $L$-degree and $x \in \Delta^1_n$, some $n$.

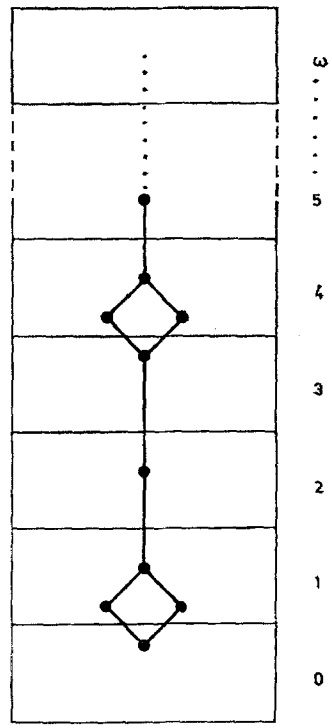



*Fig. 3*

T 3109 (JENSEN). Con(ZF) implies Con(ZF + AC + every constructible real is $\Delta^1_3$ + three is a non-constructible $\Delta^1_3$ real).

$+$ P 3110. Find a model in which the constructible reals are precisely the $\Delta^1_3$ reals, (or the $\Delta^1_n$, some fixed $n$; or precisely the second-order definable reals).

P 3111. Find a model of

$$\mathrm{ZF} + \mathrm{AC} + \bigwedge x,\ y \subseteq \omega(x \leq_L y \to x\Delta^1_3 \text{ in } y).$$

P 3112. Find a model of ZF + AC + the set of second-order definable reals is second-order definable.

T 3113 (ADDISON). ZF + $V = L \vdash$ The set of second-order definable reals is not second order-definable.

T 3114 (ADDISON). The predicate "$x \leq_L y$" is $\Sigma^1_2$.

Thus the set of constructible reals is $\Sigma^1_2$; by T 3101, if it is also $\Pi^1_2$, then every real is constructible.

## § 2. *Perfect Sets*

There is a method of coding $\mathbf{\Sigma}^1_2$ sets by reals (see SOLOVAY [10], [11] for a description) which is useful in stating criteria for a set of reals to contain a perfect subset.

+ T 3200. ZF $\vdash$ A $\mathbf{\Sigma}^1_1$ set with an element not $\Delta^1_1$ in its code contains a perfect subset, and is therefore of power $2^{\aleph_0}$.

As particular cases T 3200 we have the classical results that every Borel or analytic ($\mathbf{\Sigma}^1_1$) set of reals is either countable or contains a perfect subset, and then is of cardinality $2^{\aleph_0}$ The same need not be true for $\mathbf{\Pi}^1_1$ sets.

*T* 3201 (GÖDEL). ZF + $V = L \vdash$ There is an uncountable $\Pi^1_1$ set with no perfect subset.

More generally, by T 3000 and T3114 (cf.[H13]).

+ T 3202. ZF $\vdash \omega_1^L = \omega_1 \wedge 2^{\aleph_0} > \omega_1 \rightarrow$ there is a $\Pi^1_1$ set of cardinality $\aleph_1$ wit therefore no perfect subset.

Every $\pi^1_1$ set is the union of $\aleph_1$ Borel sets, and therefore has cardinality $\leq \aleph_1$ or is of cardinality $2^{\aleph_0}$.

T 3203 (LÉVY). Let $M$ be a c.s.m. of ZF + $V = L$; let $\varkappa \in M$, $\mathrm{cf}(\varkappa) > \omega$. Then there is an extension $N$ of $M$ with $O^N_n = O^M_n$, and in $N$, ZF + AC are true $2^{\aleph_0} = \varkappa$, $\omega_1^L = \omega_1$, and every set of reals which is ordinally definable from a real (in particular every projective set of reals) has cardinality $\varkappa$ or $\leq \omega_1$.

T 3204 (LÉVY). Let $M$ be a c.s.m. of ZF + $V = L$ + there is an inaccessible cardinal. Let $\varkappa \in M$ be a cardinal of cofinality at least $\vartheta$, where $\vartheta$ is the first inaccessible in $M$. Then there is an extension $N$ of $M$ which is a model of ZF + + AC, with the same ordinals as $M$ in which $2^{\aleph_0} = \varkappa$ and every set of reals ordinally definable from a real has cardinality $\varkappa$ or $\leq \omega$.

Note that the hypothesis of an inaccessible cannot be dropped, for

+ T 3205. ZF + AC $\vdash$ If every $\mathbf{\Pi}^1_1$ set of reals has cardinality $2^{\aleph_0}$ or $\leq \omega$, and $\omega_1 < 2^{\aleph_0}$, then $\omega_1$ is inaccessible in $L$.

PROOF. $\omega_1$ is regular in $L$, being regular. Suppose in $L$, $\omega_1 = \lambda^+$: there is a real $a$ which codes a well-ordering of $\omega$ of type $\lambda$. In $L[a]$, $\lambda$ is countable, and $\omega_1 = \omega_1^{L[a]}$. The set of reals in $L[a]$ is $\Sigma^1_2$ in $a$, and so $\mathbf{\Sigma}^1_2$, but is of cardinality $\aleph_1^{L[a]} = \aleph_1$. Using T 3000, we obtain a $\mathbf{\Sigma}^1_1$ set of reals of cardinality $\aleph_1$.

The next theorem (most of which was proved independently by Solovay) parallels T 3200.

T 3206 (MANSFIELD). ZF $\vdash$ Let $A$ be a $\mathbf{\Sigma}^1_1$ set with code $t$: if $A$ has a member not constructible from $t$, then $A$ contains a perfect subset with code $\Delta^1_2$ in $t$.

+ T 3207 (MANSFIELD; SOLOVAY). ZF $\vdash$ The following are equivalent:

(i) every uncountable $\mathbf{\Sigma}^1_2$ set has a perfect subset;

(ii) every uncountable $\mathbf{\Pi}^1_1$ set has a perfect subset;

(iii) for each real $t$ only countably many reals are constructible from $t$.

Condition (iii) is a consequence of AC + $\forall\varkappa(\varkappa \to (\omega_1)^{<\omega})$: Thus

T 3208 (SOLOVAY). ZF + AC + $\forall\varkappa(\varkappa \to (\omega_1)^{<\omega}$ $\vdash$ Every uncountable $\mathbf{\Pi}^1_1$ set has a perfect subset.

P 3209 (SOLOVAY). Show that ZF + AC + SuCC $\vdash$ Every projective set of reals has a perfect subset.

Let us contrast Mansfield's result with the following theorem of FRIEDMAN:

T 3210. ZF $\vdash$ There is an infinite $\Pi^1_1$ set of reals, every member of which except one is $\in \Delta^1_2$.

SOLOVAY points out in his paper [B11] on $\mathbf{\Sigma}^1_2$ sets of reals that using T 2017, one can show

T 3211. Con(ZF + AC + MC) implies

$$\text{Con}(\text{ZF} + \text{MC} + \text{AC} + 2^{\aleph_0} = \aleph_1 + \text{there is a } \Pi^1_2 \text{ set of reals with no perfect subset})$$

and

Con(ZF + MC + AC + $2^{\aleph_0} > \aleph_1$ + there is a $\Pi^1_2$ set of reals of cardinality $\aleph_1$).

T 3212 (LÉVY). Con(ZF) implies Con(ZF + GCH + every well ordering of reals which is ordinal definable without real parameters is denumerable or finite).

T 3213 (LÉVY). Con(ZF + $V = L$ + there is an inaccessible cardinal) implies Con(ZF + GCH + every projective well ordering (more generally, every well ordering ordinal definable from a real) of reals is finite or denumerable)).

P 3214. Find a model of ZF + AC in which $2^{\aleph_0} > \aleph_1$ and there is a projective well ordering of the continuum.

+ T 3215 (MANSFIELD). ZF + Let $A$ be a $\mathbf{\Sigma}^1_2$ well ordering. Then every real in the domain of $A$ is constructible from a code for $A$.

D 3216. Write $P(\varkappa)$ for "Every subset of $\varkappa \times \varkappa$ is in the $\sigma$-field generated by all rectangles".

T 3217 (KUNEN; SILVER). ZF + AC $\vdash P(\omega_1)$.

T 3218 (SILVER). ZF + AC + (*) $\vdash P(2^{\aleph_0})$.

T 3219 (KUNEN). ZF + AC $\vdash P(\varkappa) \to \varkappa \leq 2^{\aleph_0} \wedge \varkappa$ not RVM.

T 3220 (KUNEN; SILVER). Con(ZF + AC + MC) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \forall \varkappa(2^{\aleph_0} = \varkappa \wedge P(\varkappa) \wedge i(\varkappa, \aleph_1))).$$

T 3221 (KUNEN). In any Cohen generic or measure algebra extension of a model of ZF + AC + $2^{\aleph_0} = \aleph_1$, $P(\aleph_2)$ is false. Hence

$$\mathrm{Con}(\mathrm{ZF}) \text{ implies } \mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + 2^{\aleph_0} = \aleph_2 + \neg P(2^{\aleph_0})).$$

+ T 3222 (MANSFIELD). ZF + AC $\vdash$ The universal analytic subset of $N^N \times N^N$ does not lie in the $\sigma$-field generated by the analytic rectangles.

T 3222 answers a problem of Ulam.

P 3223 (MANSFIELD). Show that Con(ZF) implies Con(ZF + AC + the universal analytic set is not in the $\sigma$-field generated by all rectangles).

T 3224 (MARTIN, SOLOVAY). ZF + AC + $\forall \varkappa(\varkappa \to (\varkappa)^{<\omega}) \vdash$ Every $\Sigma^1_3$ set of reals has a $\Delta^1_4$ element.

T 3225 (SOLOVAY). ZF + AC + (*) $\vdash$ Every set of reals of power at most $\aleph_1$ is $\mathbf{\Pi}^1_1$.

T 3225 is proved by using Solovay's trick.

Let $\varkappa$ be 2VM, $\mu$ a normal measure.

T 3226 (SILVER). ZF + $V = L^{\mu} \vdash$ There is a $\Delta^1_3$ well ordering $\prec$ of the continuum, which has the property that the quantifier $\wedge y \preceq x$ applied to a $\Delta^1_3$ formula yields a $\Delta^1_3$ formula.

T 3027 is an immediate corollary; more generally,

T 3227 ZF + $V = L^{\mu} \vdash \wedge\, n > 3$ Unif($\mathbf{\Sigma}^1_n$).

## § 3. *Measurability and the Property of Baire*

We begin by stating what is known about the set of constructible reals ('S' for short) in various models.

T 3300 (ADDISON). ZF $\vdash$ S $\in \Sigma^1$. If there are two $I$-degrees, then by T 3114, $S \in \mathbf{\Pi}^1_2$, for then

$$S(\omega)/S = \{x \mid a \leq^L x\} \in \Sigma^1_2,$$

where $a$ is any non-constructible real.

T 3301 (PŘÍKRY). In Sacks' model D 1109, $S \in \Sigma^1_2 \cap \mathbf{\Pi}^1_2$, and is neither LM nor PB.

The next two theorems were proved independently by Kunen; see also BUKOVSKÝ [H26] and VOPĚNKA—HRBÁČEK [H47]. Recall D 1101, D 1102, and the remark after T 3010.

+ T 3302 (SOLOVAY). In $V^{\mathbf{B}_1}$, $S$ is of measure 0 and does not have the property of Baire.

T 3303 (SOLOVAY). In $V^{\mathbf{B}_2}$, $S$ is of inner measure 0, outer measure 1, and is of first category.

T 3304. ZF $\vdash$ Every $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ set is LM and PB.

T 3305 (GÖDEL). ZF + $V = L \vdash$ There is a $\Delta^1_1$ set of relas which is neither LM nor PB.

+ T 3306 (SOLOVAY). Con(ZF + AC + there is a strongly inaccessible cardinal) implies Con(ZF + AC + GCH + every uncountable set of reals which is ordinally definable from a real is LM, PB and contains perfect subset).

+ T 3307 (SOLOVAY). Con(ZF + AC + there is a strongly inaccesible cardinal) implies Con(ZF + $DC^\omega$ + every uncountable set of reals is LM and PB and contains a perfect subset.)

Solovay proved T 3306 by examining the model constructed by Lévy to prove T 3204. T 3307 follows by considering the submodel of all sets hereditarily ordinally-definable-from-a-real.

T 3308 (HÁJEK [H39]). Con(ZF + AC + there is a strongly inaccessible cardinal) implies Con(ZF + there is no choice set for the Lebesgue decomposition of the real line + $\omega_1$ is regular + there is an uncountable set of reals without a perfect subset).

+ P 3309. The hypothesis of T 3307 cannot be weakened to Con(ZF): but does Con(ZF) imply Cpn(ZF + $DC^\omega$ + every uncountable set of reals in LM)?

P 3310. ZF + $DC^\omega$ + every set of reals is LM $\vdash \aleph_0$ adj $2^{\aleph_0}$?

T 3311 (SOLOVAY). Con(ZF) implies Con(ZF + GCH + there is a translation invariant extension of Lebesgue measure defined (at least) on all sets of reals ordinal definable from a real).

T 3312 (SOLOVAY). Con(ZF) implies Con(ZF + $\bigwedge \alpha C^{\alpha}$ + not every set of reals in Lebesgue measurable but there is a translation invariant extension of Lebesgue measure defined on all sets of reals).

T 3313 (SOLOVAY). Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} > \aleph_1$ + the real line is the union of $\aleph_1$ sets of measure 0).

For the next three theorems, let $M$ be a c.s.m. fo ZF + $V = L$; and let $\varkappa$ be an infinite regular cardinal in $M$. (Theorems of this sort have also been proved by KUNEN; for T 3316 see BUKOVSKÝ [H 26] and VOPĚNKA—HRBÁČEK [H47].)

T 3314 (MARTIN, SOLOVAY). There is an extension $N$ of $M$ with $\mathrm{Card}^N = \mathrm{Card}^M$ in which ZF + AC is true, $2^{\aleph_0} = \varkappa$, the ideal of sets of reals of measure 0 is $\varkappa$-complete and the ideal of sets of reals of 1st category is $\varkappa$-complete.

+ T 3315 (SOLOVAY). There is an extension of $M$ with the same cardinals as $M$ which is a model of ZF + AC in which $2^{\aleph_0} = \varkappa$, every set of reals of cardinality $< \varkappa$ is of 1st category, but there is a non-measurable set of cardinality $\aleph_1$.

T 3316 (SOLOVAY). There is an extension of $M$ with the same cardinals as $M$ which is a model of ZF + AC in which $2^{\aleph_0} = \varkappa$, every set of reals of cardinality $< \varkappa$ is of measure 0, but there is a set of reals of cardinality $\aleph_1$ which is not PB.

D 3317. A set $A$ of reals has *strongly measure* 0 iff for any sequence of positive reals $\{a_n\}_{n \in \omega}$ there is a sequence of open intervals $\{I_n\}$ such that $A \subseteq \bigcup I_n$ and $\mu(I_n) < a_n$.

+ P 3318. Find a c.s.m. of ZF + AC + "every set of reals strongly of measure 0 is countable (Borel's conjecture)", given a model of ZF + $V = L$.

Gödel's theorem T 3305 should be contrasted with

T 3319 (SOLOVAY). ZF + AC $\vdash$ Let $x \subseteq \omega$ and $\omega_1^{L[x]} < \omega_1$. Then every set of reals $\Sigma_2^1$ in $x$ is LM and PB.

which is proved using Shoenfield's theorem T 3101 and some very fortunate properties of random reals, and has the immediate corollary

T 3320 (SOLOVAY). ZF + AC + $\bigvee \varkappa(\varkappa \to (\omega_1)^{<\omega})$ $\vdash$ Every $\mathbf{\Sigma}_2^1$ set of reals is LM and PB.

+ T 3321 (cf T 3314). ZF + AC + (*) $\vdash$ The ideal of sets of measure 0 and the ideal of sets of first category are both $\aleph_2$-additive; consequently every $\mathbf{\Sigma}_2^1$ set of reals is LM and PB.

+ T 3322 (TANAKA [13]). ZF $\vdash$ The Lebesgue measure of a $\Sigma_1^1$ set of reals is a $\Sigma_1^1$ real.

The following answers a question of Příkry:

T 3323 (KUNEN; SOLOVAY). Let $M$ $\varkappa$ be as in T 3314. Extend $M$ by the regular open algebra of $2^{\varkappa}$ (viz. add $\varkappa$ generic reals). Then in the extension $2^{\aleph_0} = \varkappa$ and there is a $\Delta^1_2$ nonmeasurable set.

P 3324 (SOLOVAY). Show that ZF + AC + SuCC $\vdash$ Every projective set of reals is LM.

We end this paragraph with some results on RVM'S.

+ T 3325 (KUNEN). ZF + AC $\vdash$ Let $\varkappa$ be RVM. Then

(i) there is a set of reals of power $< \varkappa$ which is not LM;

(ii) every set of reals of power $< \varkappa$ is of first category;

(iii) the real line is the union of fewer than $\varkappa$ sets of first category; but

(iv) the real line is not the the union of fewer than $\varkappa$ sets of Lebesgue measure 0.

## § 4. *Some problems of Hausdorff*

D 3400 Write $H(\varkappa)$ for "There is a family $F$, $\overline{\overline{F}} = \varkappa^+$, of functions $f: \varkappa \to \varkappa$ such that given any $g: \eta \to \varkappa$ there is an $f \in F$ such that $\bigwedge \nu < \varkappa g(\nu) < f(\nu)$.

Hausdorff asked whether, even if the continuum hypothesis failed, it would still be possible to prove $H(\omega)$. Now

$$\mathrm{ZF} \vdash H(\omega) \leftrightarrow$$

$$\leftrightarrow \bigvee \mathfrak{F} \subseteq N^N \big(\overline{\overline{\mathfrak{F}}} = \aleph_1 \text{ and } \bigwedge f \in N^N \bigvee g \in \mathfrak{F} \bigvee m > \omega \bigwedge n > m (g(n) > (f(m)))\big):$$

that is, $H(\omega)$ holds iff there is a family of functions of cardinality $\aleph_1$ such that every function is eventually majorized by a function in the family.

T 3401 (SOLOVAY). Let $M$ be a c.s.m. of ZF + $V = L$. Let $N_1$ be obtained be adding $\aleph_2$ generic reals to $M$; then

$$N_1 \models \mathrm{AC} \wedge 2^{\aleph_0} = \aleph_2 \wedge \neg H(\omega).$$

Let $N_2$ be obtained by adding $\varkappa$ random reals, where in $M$, $\omega_1 < \varkappa = \varkappa = \wedge\, \mathrm{cf}(\varkappa) > \omega$. Then

$$N_2 \models \mathrm{AC} \wedge 2^{\aleph_0} = \varkappa > \aleph_1 \wedge H(\omega).$$

Indeed, the set of constructible functions $\in N^N$ will serve as the required $\mathfrak{F}$.

D 3402. A set $X$ of reals has *Lusin's property* (LP) iff

$$\overline{\overline{X}} = 2^{\aleph_0} \text{ and } \overline{\overline{X \cap Y}} \leq \aleph_0 \text{ for every } Y \text{ of first category.}$$

The existence of a set with LP implies (in ZF) that $H(\omega)$ is false. The first part of T 3401 has been strengthened to

T 3403 (VOPĚNKA, HRBÁČEK [H47]). Let $M$ be as in T 3401, and let $N$ be obtained by adding $\varkappa$ generic reals, where $\mathrm{cf}(\varkappa) > \omega$. Then there is in $N$ a set with LP.

Jensen noticed that models for ZF + AC + $2^{\aleph} > \varkappa^+ + H(\omega)$ where $\varkappa$ is inaccessible in $M$ (or $\varkappa = \omega$), could be constructed using Sacks' forcing. Silver then remarked that by collapsing cardinals underneath the inaccessible $\varkappa$ one could get (say) a model of ZF + AC + $2^{\aleph_0} > \aleph_2 + H(\aleph_1)$. Hence

T 3404 (JENSEN, SILVER). Con(ZF + $V = L$ + $\bigvee \varkappa$ $\varkappa$ inaccessible) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + 2^{\aleph_1} > \aleph_2 + H(\aleph_1)).$$

P 3405. Does the conclusion of 3404 follow from Con(ZF)?

There is some relation between $H$ and the property $F$ discussed earlier (D 1126): by using Sacks' forcing one may show

T 3406. Con(ZF) implies

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \neg\,\mathrm{CH} + F(\aleph_0)).$$

For $f \in N^N$, $g \in N^N$, define $f \prec g$ to hold iff $\bigvee m \bigwedge n < m(f(n) < g(n))$. $\prec$ has been investigated by HECHLER [*] who shows

T 3407 (HECHLER). Let $M$ be a c.s.m. of ZF + AC and let $\langle S, <_S\rangle$ be a partially ordered set in $M$ with $S \leq 2^{\aleph_0}$ in $M$; then there is an $N \supseteq M$ which is a model of ZF + AC, $\mathrm{Card}^N = \mathrm{Card}^M$, for all cardinals $\varkappa$, $(2^\varkappa)^N = (2^\varkappa)^M$ and $\langle S, <_S\rangle$ is embeddable in $\langle N^N, \prec\rangle$.

T 3408 (HAUSDORFF). ZF + AC $\vdash$ The unit interval is the union of a properly increasing (nested) sequence of $G_\delta$ sets of length $\aleph_1$, but is not the union of a properly increasing sequence of sets in $G_\delta \cap F_\sigma$.

T 3409 (SOLOVAY). Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} > \aleph_1$ + the unit interval is the union of a properly increasing nested sequence of $F_\sigma$ sets of length $\aleph_1$.

(The last property is trivially true if $2^{\aleph_0} = \aleph_1$, as every countable set is an $F_\sigma$).

T 3410 (SOLOVAY). Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} > \aleph_1$ + the unit interval is not the union of a properly increasing sequence of $F_\sigma$ sets of length $\aleph_1$).

+ P 3411 (KEISLER). Let $\varkappa$ be an uncountable cardinal. Does there exist an ultrafilter $D$ on the set $\varkappa$ such that

(i) every $x \in D$ is of power $\varkappa$;

(ii) for every set $E \subseteq D$ of power $\varkappa$, some element of $\varkappa$ belongs to infinitely many elements of $E$?

The case $\varkappa = \aleph_1$ is particularly interesting. Silver has shown that the answer is yes if $\varkappa$ is RVM.

P 3412 (Reyes). Let $R = \{\langle P_1, P_2, \ldots\rangle : P_i \subseteq \omega\} = (2^\omega)^\omega$. $S \subseteq R$ is *invariant* iff, for every permutation $\pi$ of $\omega$, $(P_1, \ldots\rangle \in S$ iff $\langle \pi'' P_1, \ldots\rangle \in S$. Is there an invariant $S \subseteq R$ which is neither meagre nor comeagre in the product topology?

The answer is yes if $2^{\aleph_0} = \aleph_1$ or (remarked by Solovay) if $\neg\, i(2^{\aleph_0}, \aleph_1)$.

## Section 4. Kurepa's Conjecture and results in Model Theory

### § 0. *Conjectures of Kurepa and Chang*

+ D 4000 (Kurepa). $\aleph_\alpha$ has Kurepa's property (written $K(\aleph_\alpha)$) iff there is a family $S$ with $S = \aleph_{\alpha+2}$ of subsets of $\aleph_{\alpha+1}$ such that for any $\beta < \aleph_{\alpha+1}$, $\{X \cap \beta \mid X \in S\} \leq \aleph_\alpha$.

D 4001. (ZF + AC) $\langle \varkappa, \lambda\rangle = >_0 \langle \varkappa', \lambda'\rangle$ iff any theory with a model of type $\langle \varkappa, \lambda\rangle$ has a model of type $\langle \varkappa', \lambda'\rangle$, where $\varkappa, \lambda, \varkappa', \lambda'$ are infinite cardinals, $\varkappa \geq \lambda$, $\varkappa' \geq \lambda'$, and "type $\langle \varkappa, \lambda\rangle$" is as defined in Vaught [14].

D 4002. (ZF + AC) $\langle \varkappa, \lambda\rangle = <_1 \langle \varkappa', \lambda'\rangle$ iff any relational structure $\langle A, U, R\rangle$ where $\overline{\overline{A}} = \varkappa$, $\overline{\overline{U}} = \lambda$, $U \subseteq A$ and $R \subseteq A$ has an elementary substructure $\langle B, V, S\rangle$ where $\overline{\overline{B}} = \varkappa'$, $\overline{\overline{V}} = \lambda'$, $V \subseteq B$, and $S \subseteq B$.

T 4003 (Lévy, Rowbottom; Bukovský [H45]). Con(ZF + $V = L$ + there is an inaccessible cardinal) implies Con(ZF + GCH + $K(\aleph_0)$.

+ T 4004 (Silver [*]). Con(ZF + $V = L$ + there is an inaccessible cardinal) implies Con(ZF + GCH + $\neg\, K(\aleph_0)$ + $K(\aleph_1)$).

T 4005 (Vaught). ZF + AC $\vdash \bigwedge \alpha(\langle \aleph_{\alpha+1}, \aleph_\alpha\rangle = >_0 \langle \aleph_1, \aleph_0\rangle)$

+ T 4006 (Chang). $\bigwedge \alpha\beta(\aleph_\beta$ regular $\rightarrow . \langle \aleph_{\alpha+1}, \aleph_\alpha\rangle = >_0 \langle \aleph_{\beta+1}, \aleph_\beta\rangle)$.

Now in [14], which discusses the property $=>_0$ in greater detail, Vaught remarks that there is a sentence $A$ in the first-order language with a distinguished unary predicate such that for all $\alpha$, $A$ has a model of type $\langle \aleph_{\alpha+k}, \aleph_\alpha\rangle$ iff $K(\aleph_\alpha)$. For further theorems and problems see Silver [*].

+ D 4007. (ZF + AC) $\varkappa$ is a Rowbottom cardinal ($R(\varkappa)$) iff

$$\varkappa = \varkappa < \omega \,.\wedge.\, \bigwedge \lambda(\omega \leq \lambda < \varkappa \rightarrow . \langle \varkappa, \lambda\rangle = >_1 \langle \varkappa, \omega\rangle) .$$

Chang's conjecture (CC) is the assertion that $\langle \aleph_2, \aleph_1 \rangle =<_1 \langle \aleph_1, \aleph_0 \rangle$. Let us denote by $Q$ the assertion that there is a family $F$ of functions $f: \omega_1 \to \omega$ with $|F| = \omega_2$ and

$$\bigwedge f, g \in G \bigvee \alpha < \omega_1 \bigwedge \beta (\alpha < \beta < \omega_1 \to f(\beta) \neq (\beta)).$$

T 4008 (SILVER). ZF + AC $\vdash$ CC $\to \neg$ $Q$.

T 4009 (ROWBOTTOM). ZF + AC $\vdash \bigwedge \varkappa (\varkappa\ 2\text{VM} \to R(\varkappa))$.

T 4010 (PŘÍKRY). ZF + AC $\vdash \bigwedge n < \omega (\varkappa_n 2\text{VM} \to R(\bigcup_{n<\omega} \varkappa))$.

T 4011 (ROWBOTTOM). ZF + AC $W$ CC $\to V \neq L$.

T 4012 (KEISLER, ROWBOTTOM). ZF + AC + CC $\vdash$ "There are several weakly compact cardinals" is true in $L$.

+ T 4013 (SILVER). Con(ZF + AC + MC) implies Con(ZF + GCH + CC).

T 4014 (SILVER, using T 2118). Con(ZF + AC + MC) implies

$$\text{Con}(\text{ZF} + \text{GCH} + \langle \aleph_1, \aleph_0 \rangle =>_0 \langle \aleph_{\omega+\alpha}, \aleph_\omega \rangle).$$

P 4015. Does ZF + AC + $Q \vdash K(\aleph_0)$?

P 4016. Is $Q$ independent of something not very big?

### + § 1. *End Extensions*

D 4100. (ZF + AC) $\langle B, F \rangle$ is an end extension of $\langle V_\alpha, \in \rangle$ iff it is a proper elementary extension and $bFa \wedge a \in V_\alpha .\to b \in V_\alpha$.

P 4101 (KEISLER). Is there an inaccessible $\alpha$ such that $\langle V_\alpha, \in \rangle$ has an end extension but no well-founded end extension?

P 4012 (KEISLER). Let $\varkappa$ be the first $\pi^1_1$ indescribable cardinal. Has $\langle V_\varkappa, \in \rangle$ end extensions of arbitrarily large cardinality, or even of cardinality $\varkappa^{++}$?

## Section 5. Other systems

### + § 0. *Variations of Zermelo—Fraenkel Set Theory*

By NBG we mean the axioms of blooks $A$ to $D$ of Gödel's monograph.
T 5000. Con(ZF) implies and is implied by Con(NBG).

T 5001. Any theorem of NBG in which only set variables occur is a theorem of ZF, and vice versa.

Let us note two facts about axiom $E$, the class form of AC:

T 5002 (EASTON). Con(NBG) implies Con(NBG + GCH + $\neg$ $E$).

T 5003 (SOLOVAY). Any theorem of NBG + $E$ in which only set variables occur is provable in ZF + AC.

Easton indeed showed more than T 5002, namely that if NBG is consistent, then so is NBG + GCH $\vdash$ "There is no class selecting one element from every unordered pair in the universe."

Let ZFU be the usual modification of ZF allowing individuals. The article of LÉVY [5] gives all but the most recent results. A theorem of GAUGHAN ([9], Theorem 11.5.9) establishes the existence of a group of the sort needed to answer affirmatively the problem mentioned on the last page of [5].

D 5004. We write $C(S)$ $(C^{\mathrm{wo}}(S))$ for the conjunction of the axioms $C_n(C^{\mathrm{wo}})$ for $n \in S$, where $S$ is a finite subset of $\omega$.

T 5005 (GAUNTT). Con(ZF) implies $\mathrm{C}_n^{\mathrm{wo}}(\mathrm{ZFU} + \bigwedge n(1 < n \in \omega \to \neg C_n)$.

T 5006 (GAUNTT).

$$\mathrm{ZFU} \vdash \bigwedge n, m \in \omega(C_{n+m} \to . C \vee C^{\mathrm{wo}}_{n+m});$$

$$\bigwedge n, m \in \omega(C^{\mathrm{wo}}_{n+m} \to . C^{\mathrm{wo}}_n \vee C^{\mathrm{wo}}_m).$$

P 5007 (GAUNTT). Does ZFU $\vdash \bigwedge n, m \in \omega(C_{n+m} \to . C \vee C_m)$?

Let $m \in \omega$ and $S$ a finite subset of $\omega$.

T 5008 (GAUNTT). If ZFU $\vdash C^{\mathrm{wo}}(S) \to C^{\mathrm{wo}}_m$, then ZFU $\vdash C(S) \to C_m$.

T 5009 (GAUNTT). UFU $\vdash C^{\mathrm{wx}}(S) \to C^{\mathrm{wx}}_m$ iff for subgroup $G$ of Sym($m$) without fixed points there exist finitely many proper subgroups $H_1, \ldots, H_k$ of $G$ such that $|G: H_1| + \ldots + |G: H_k| \in S$.

T 5009 gives a decision procedure for the relative consistency of ZFU + + $C^{\mathrm{wo}}(S) + \neg C^{\mathrm{wo}}_m$; a procedure for ZFU + $C(S) + \neg C_m$ has yet to be found.

There is no general method for converting a proof of independence from ZFU to one from ZF, for consider the axiom of foundations itself, or less trivially the assertion $X$ that every set which can be linearly ordered can also be well-ordered: ZF $\vdash X \to$ AC, but if Con(ZFU) then Con(ZFU + $X$ + + $\neg$ AC). However some independence proofs are convertible:

+ T 5010 (JECH, SOCHOR [H37, H38]). Let a sentence $A$ have one of the following forms:

$\bigvee a \bigwedge x \in a B(x, a)$

$\bigvee a \bigwedge x \in S(a) G(x, a)$

$\bigvee a \bigwedge x \in S(S(a))\, B(x, a)$,

etc.,

where $B$ is a restricted ($\Delta_0$) formula. Then

$$\mathrm{Con}(\mathrm{ZFU} + A) \text{ implies } \mathrm{Con}(\mathrm{ZF} + A).$$

Let $Z$ denote Zermelo set theory, $D$ the axiom of foundations.

P 5011. Find a sentence $A$ of set theory such that $Z - D + A$ is consistent and has among its theorems all the axioms of $\mathrm{ZF} - D$.

P 5012 (LÉVY). Let $T$ be the theory $\mathrm{ZF} - D + \mathrm{AC}$. For each formula $A(x, y)$ of set theory (without parameters) add a new unary operation symbol $f_A$ and the following axiom schemata:

S1. If $A(x, y)$ is an equivalence relation, then for all $x, y$

$$f_A(x) = f_A(y) \leftrightarrow A(x, y).$$

S2. All instances of the axiom schema of replacement which contain the new symbols.

Call the extended theory $S$. Is there a sentence $A$ of ZF (that is, without the new symbols) which is a theorem of $S$ but not of $T$?

T 5013 (HÁJEK [H14]). $\mathrm{NBG} - D \vdash$ Let $R$ be a regular extensional relation. Then there is a model relation $S \supseteq R$ such that

(1) $S''\{x\} = R'\{x\}$ for all $x$ in the field of $R$;

(2) $U \subseteq \mathrm{Fld}(R) \rightarrow \bigvee Z(u = S''\{Z\})$.

+ § 1. *Variations on Quine's Set Theory*

Let NFU be the system NF with extensionality replaced by

$$\bigvee z \in x \wedge \bigwedge z(z \in x \rightarrow z \in y) .\rightarrow x = y.$$

Let $T$ be ordinary type theory with $\omega$ levels. Let Inf be the axiom of infinity in the form "There is an ordering which is not a well-ordering", and AC the axiom of choice in the form "Every disjoint set of nonempty sets has a choice set".

+ T 5100 (JENSEN). Con($T$) implies Con(NFU).

T 5101 (JENSEN). Con($T$ + Inf + AC) implies Con(NFU + Inf + AC).

T 5102 (JENSEN). Con($T$ + Inf + $\neg$ AC) implies Con(NFU + Inf + $\neg$ AC).

These results and the consistency of $T$ can be proved in elementary number theory. Hence

T 5104 (JENSEN). ENT $\vdash$ Con(NFU).

Since NFU + Inf $\vdash$ Con(ENT), Inf is independent of NFU, in contradiction to the following result:

T 5105 (SPECKER). NF $\vdash$ Inf $\wedge \neg$ AC.

Let NFU$^+$ be NFU augmented by the following induction schema:

$$(+) \qquad A(0) \wedge \bigwedge n(A(n) \rightarrow A(n+1)) . \rightarrow \bigwedge nA,$$

where the variable $n$ ranges over the finite numbers, which are defined as equivalence classes of (Russell) finite sets.

T 5106 (HAILPERIN). NF is finitely axiomatizable.

An extension of Hailperin's argument proves

T 5107 (JENSEN). NFU is finitely axiomatizable.

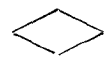
By T 5107 and a theorem of Kreisel it follows that (+) is independent of NFU.

P 5108 (JENSEN). Is there a reasonable set theory $S$ such that the equiconsistency of $S$ and NFU$^+$ is provable in elementary number theory?

T 5109 (JENSEN). ZF $\vdash$ Con(NFU$^+$).

T 5109 is a consequence of the relativization to $L$ of the following theorem.

T 5110 (JENSEN). ZF + AC $\vdash$ If $\alpha$ is any ordinal, there is a model of NF containing $V_\alpha$ as an initial segment.



## NOTES ON SUBSEQUENT DEVELOPMENTS

I have not in these notes sought to be exhaustive, but have largely confined myself to mentioning general areas of development, with suggestions for further reading.

Accounts of many of the results marked [*] in the original text have appeared in the *Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society (held at the University of California, Los Angeles, California, from July 10th to August 5th, 1967)*, published as Parts I and II of Volume 13 of the series of Proceedings of Symposia of the Soociety under the title *Axiomatic Set Theory*. The two parts, the first edited by SCOTT and the second by JECH, are called here [*, I] and [*, II] respectively.

Of the original list of references, [B10] has been published as SOLOVAY [S120]; [B11] is to be found in the *Symposium Papers commemorating the sixtieth birthday* KURT GÖDEL, edited by J. J. BULLOFF, T. C. HOLYOKE and S. W. HAHN, and published by the Springer-Verlag in 1969; and [B15] will now appear as KANAMORI, REINHARDT and SOLOVAY [S62]. [B46] has not been published, but much of the material therein may be found in DRAKE [S22].

## Section 1.

### § 1.

Many results in this paragraph are proved in JECH [S45].

D 1104. The properties "generic over $L$" and "random over $L$" are $\Pi^1_2$.

T 1106. For a refinement of this theorem, see JENSEN [S51] who exhibits a notion of forcing in $L$ such that the property of being Jensen-generic over $L$ is $\Pi^1_2$, and if $a$ is Jensen-generic over $L$, $a$ is the only such real in $L[a]$, and is accordingly $\Delta^1_3$ in $L[a]$. In contrast, if there is one real generic over $L$ there is a perfect set of them.

T 1108 is false as stated: add the condition that for all $f: \omega \to 2$ $\bigcap_n T_{f\restriction n}$ is a singleton.

D 1111. Silver reals are a special case of reals studied by GRIGORIEFF [S34], who amends D 1110 by fixing a non-principal ideal $I$ on $\omega$, and requiring that $A \cup B \in I$. He shows that the resulting generic real is of minimal degree iff $I$ is the ideal dual to a Ramsey ultrafilter. Silver's extension is the result of first adding a Ramsey ultrafilter generically (without adding new reals: the partial ordering used is that of the infinite subsets of $\omega$ modulo finite differences) and then using it to make a Grigorieff extension.

T 1113 follows easily from Grigorieff's work: if $a$ is Silver over $L$ then

$$\{x \mid x \in L \cap S(\omega) \wedge x \cap a \notin L\}$$

is a Ramsey ultrafilter $F$ such that $'a \notin L[F]$, $F \notin L$, $S(\omega) \cap L[F] \subseteq L$ and $a$ is Grigorieff with respect to $F$ over $L[F]$.

D 1117. In MATHIAS [S89] a notion of generic real is studied which has the feature that if $x$ is generic then every infinite subset of $x$ is also generic; moreover two disjoint infinite subsets of $x$ will always be of incomparable $L$-degrees.

In this connection, see SOLOVAY [S119].

P 1128. Yes (KUNEN; BAUMGARTNER).

T 1130. See JENSEN and SOLOVAY [S59] for applications of Solovay's trick, which has also been developed by JENSEN in an important paper [S52] from which we quote Theorem 1: Let $\langle M, A\rangle$ be a transitive model of ZF + GCH + $+ \neg O^\#$ (the last being the statement that $O^\#$ does not exist (cf D 2037)), where $A \subseteq On \cap M$. Then there is an $\langle M, A\rangle$ definable class of conditions **P** such that if $G$ is U-generic over $\langle M, A\rangle$ and $N = M[G]$, then

(a) $N$ is a model of ZF + GCH + $\neg O^\#$;

(b) cardinals and cofinalities are preserved, as are the large cardinal properties: Mahlo, weakly compact, ineffable;

(c) there is a real $a \subseteq \omega$ such that $a \in N$, $N \models V = L[a]$ and $A$ and $G$ are $N$-definable from the parameter $a$.

The paper also answers in a negative sense made precise therein a problem of Solovay: if $b$ is a real, or more generally a set of ordinals, from which $O^{\#}$ is not constructible, is $b$ generic over $L$ with respect to a *set* of conditions.

§ 2.

P 1203. Great inroads have been made recently into the singular cardinals problem, starting with SILVER's proof [S1116]; for an elementary proof, see BAUMGARTNER and PŘÍKRY [S5]) that if $\{\zeta < \omega_1 \mid 2^{\aleph_\zeta} = \aleph_{\zeta+1}\}$ is stationary in $\omega_1$, then $2^{\aleph_{\omega_0}} = \aleph_{\omega_1+1}$. Similar results hold for every singular cardinal of uncountable cofinality. This theme has been developed by SILVER; JECH and PŘÍKRY [S50]; GALVIN and HAJNAL [S32]; JENSEN [S53]; MAGIDOR and SHELAH. We quote three results:

(SILVER) If $\varkappa$ is a singular strong limit cardinal with $2^\varkappa > \varkappa^+$ and $\lambda = \mathrm{cf}(\lambda) < \varkappa$ then the filter $\mathfrak{F}$ of closed unbounded subsets of $\lambda$ is such that $\lambda$ is measurable in $L^{\mathfrak{F}}$ (cf Theorem 5.8 of KUNEN [S71]).

(JENSEN) Assume $\neg\, O^{\#}$. Let $X$ be an uncountable set of ordinals. Then there is $Y \in L$ such that $\overline{\overline{X}} = \overline{\overline{Y}}$ and $X \subseteq Y$ ("the covering lemma").

(PŘÍKRY) If $2^\omega$ is RVM, then for all $\nu > 2^\omega$, $2^\nu = 2^\omega$.

As for the possibility of achieving a model in which $2^{\aleph_n} = \aleph_{n+1}$ for each $n < \omega$ but $2^{\aleph_\omega} = \aleph_{\omega+2}$, it follows from Jensen's covering lemma that that cannot be done in a Boolean extension of $L$, whereas MAGIDOR [S79] can do so starting from a model containing two very large cardinals.

T 1205 is due to GÖDEL.

P 1207. MAGIDOR has shown that i $\varkappa$ is $2^\varkappa$-supercompact and $\lambda$ is a regular cardinal less than $\varkappa$ then in some cardinal-preserving Boolean extension, $\mathrm{cf}(\varkappa) = \lambda$; which improves PŘÍKRY's T 2118.

T 1212. See JENSEN and SOLOVAY [S59].

T 1214. See MCALOON.

T 1215. See MCALOON [S92].

D 1216. For proofs of (improved versions of ) TT 1217—1220 see VOPĚNKA and HÁJEK [S130], BALCAR [S2], BUKOVSKÝ [S12] and later papers by Hájek;

and for further investigations of the sequence HOD, $(\mathrm{HOD})^{\mathrm{HOD}}$, ... see McAloon [S91], Jech [S46] and Grigorieff [S35].

D 1221. Souslin's problem has been extremely stimulating: see Devlin and Johnsbraten [S17], [S20]. Souslin trees also play a part in the papers by Jech [S46] and Jensen and Johnsbraten [S20]. For more on trees, see Jech [S48].

P 1228. Yes (Jensen; see [S20]).

P 1229. No (Jensen; see [S17]).

In axiomatizing and generalizing his proof, Jensen formulated several combinatorial principles, of which we give two examples:

$\diamondsuit_\varkappa$: There is a family $\langle S_\zeta \mid \zeta < \varkappa\rangle$ such that $S_\zeta \subseteq \zeta$,

and for each $X \subseteq \varkappa$, $\{\zeta | X \cap \zeta = S_\zeta\}$ is stationary in $\varkappa$.

$\square_\varkappa$: There is a sequence $\langle C_\lambda \mid \lambda < \varkappa^+ \wedge \lim(\lambda)\rangle$ such that

(i) $C_\lambda$ is closed and unbounded in $\lambda$;

(ii) $\mathrm{cf}(\lambda) < \varkappa \to |C_\lambda| < \varkappa$;

(iii) if $\gamma$ is a limit point of $C_\lambda$, then $C_\gamma = \gamma \cap C_\lambda$.

If $V = L$, then $\square_\varkappa$ holds for all infinite cardinals $\varkappa$, and $\diamondsuit_\varkappa$ for all regular $\varkappa$ (Jensen). Jensen's discussion of $\square$ in [S17] ans [S54] turns on his *fine structure theory* of $L$; Silver has devised some *machines* which enable the principles $\square_\varkappa$ to be established in $L$ without it.

P 1320. Yes.

P 1232. Ms Rudin has constructed an example of a normal Hausdorff space which is not countably paracompact in ZF + AC alone. See Rudin [S111], Chapter IX, (2).

T 1234. Using forms of $\diamondsuit$ and $\square$ Jensen has proved that if $V = L$, $\varkappa$ is weakly compact iff there is no Souslin $\varkappa$-tree. See Devlin [S17].

D 1236. The version of (*) with "at most $\aleph_1$" replaced by "fewer than $2^{\aleph_0}$" has become known as Martin's axiom, and is fully discussed in Martin and Solovay [S88] and its consistency proved in Solovay and Tennenbaum [S122]. For its use in general topology see Rudin [S111] and Tall [S123], [S124].

T 1237. See Solovay and Tennenbaum [S122].

T 1238. See Solovay and Tennenbaum [S122].

T 1240 is stated incorrectly: the hypothesis that for some real $X$, $\aleph_1 = \aleph_1^{[X]}$ should be added. The conjunction of the two statements "every subset of is constructible from a real" and "for each real $X$, $X^\#$ (D 2037 relativized) exists", both of which incidentally are consequences of the axiom of determinacy, is incompatible with the axiom of choice; so using a Ramsey cardinal one may easily find a model in which (*) and the negation of T 1240 both hold.

A refined version of T 1240, due to Solovay, is stated as Theorem 3.2 of MARTIN and SOLOVAY [S88], and is this: suppose that (*) holds and that for some real $X$, $\aleph_1 = \aleph_1^{L[X]}$. Then every set of reals of power $\aleph_1$ is $\mathbf{\Pi}^1_1$.

T 1241. FLEISSNER [S29] has proved, using a principle similar to $\diamondsuit$, that if $V = L$, then every locally compact normal Moore space is metrizable.

T 1242. For more on sequential compactness, see BOOTH [S11].

For an application of $\diamondsuit$ to group theory, see SHELAH [S113], [S114], parts of which are expounded in EKLOF [S24]; and for applications of Jensen's combinatorial principles to general topology, see RUDIN ([S111], Chapter VI), and OSTASZEWSKI [S101].

For a study of gaps in the constructible hierarchy, see MAREK and SREBRNY [S83].

§ 3.

For this whole paragraph see the books of JECH [S47] and FELGNER [S26].

T 1303. Proved in FELGNER [S26].

T 1304. Proved in FELGNER [S26].

P 1309. No (GAUNTT).

For an application of Thompson's list of minimal simple groups to these independence questions, see CONWAY [S15].

T 1319. See MATHIAS [*, II] or FELGNER [S26].

T 1320. See HALPERN, LÉVY [*, I].

P 1321. Yes (FELGNER [S27]).

P 1324. Yes (PINCUS).

T 1325 is false, the error in Felgner's proof being found by Morris, who has shown that if FP is the statement that every total ordering has a cofinal sub-well-ordering then $\mathrm{ZF} \vdash \mathrm{OE} + \mathrm{FP} \to \mathrm{AC}$, but if Con(ZF) then

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{FP} + \neg\, \mathrm{OE}).$$

P 1327. No (PINCUS [S104]).

P 1330. Such a model has been found by MORRIS [S99].

P 1331. By the covering lemma of JENSEN [S53], if $\omega_1$ and $\omega_2$ are both cofinal with $\omega$, $O^{\#}$ exists.

Magidor has, using $\zeta$ supercompact cardinals and a measurable cardinal found a model in which the first $\zeta$ uncountable alephs are cofinal with $\omega$.

P 1332. No (SAGEEV [S112]).

P 1354. If every cardinal has a strong successor, then for each infinite cardinal $\mathfrak{p}$, $2\mathfrak{p} = \mathfrak{p}$. (TARSKI; TRUSS [S126]).

## Section 2.

Further papers in this area are ASH [S1], BELL, FREMLIN [S7]; EDWARDS [S23]; GARDINER [S33]; HOWARD [S44]; JECH [*, I]; MONRO [S97]; and TRUSS [S127], [S128].

For this whole section, see DREKE's book [S22], and the expository paper by KANAMORI, REINHARDT and SOLOVAY [S62].

### § 0.

D 2001. There are no Moschovakis cardinals (MARTIN; KUNEN; NEIL WILLIAMS).

T 2003. MENAS [S93] has proved that a measurable cardinal which is a limit of strongly compact cradinals is strongly compact, and the first such cardinal is not supercompact.

T 2007. S. Thomson has computed $\varkappa(\alpha + 1)$ explicitly from $\varkappa(\alpha)$. See DRAKE [S22].

T 2010. The following strong from of weak compactness is important in studying $L$: call $\eta$ *ineffable* if given any partition $f$: $[\varkappa]^2 \to 2$, there is a stationary set $X$ with $f$ constant on $[X]^2$. See Devlins book [S17]. If $\varkappa$ is ineffable, then $\diamondsuit_\varkappa$ holds (Kunen — see JENSEN—KUNEN [S58]). The notion is studied further in Baumgartner [S3].

T 2011 is now vacuous.

T 2013 is also due, independently, to Paris: see KUNEN—PARIS [S75].

P 2014. Yes if $V = L$, by T 2018. MITCHELL [S95] has constructed a model in which a certain measurable cardinal has precisely two normal measures; the set of smaller measurable cardinals has measure 1 in one measure and measure 0 in the other, so both measures are definable.

T 2016. See KUNEN [S71].

T 2018. See KUNEN [S71], and for another study of iterations, see GAIFMAN [*, II]. For safety assume AC in the following.

Let $\mathcal{U}_0$ be a normal measure on $\kappa_0$, and let $V_{(0)} = V$. Define $V_{(1)}$ to be the transitive collapse of the ultrapower $V^{\kappa_0}/\mathcal{U}_0$: in short, $V_{(1)} = \mathrm{Ult}(V, \mathcal{U}_0)$; and let $i_{01}$ be the canonical elementary embedding $V_{(0)} \to V_{(1)}$. Let $\kappa_1 = i_{01}(\kappa_0)$ and $\mathcal{U}_1 = i_{01}(\mathcal{U}_0)$. Let $V_{(2)} = \mathrm{Ult}(V_{(1)}, \mathcal{U}_1)$, constructed using only those functions $f: \kappa_1 \to V_{(1)}$ which are in $V_{(1)}$; define $i_{12}$, $\kappa_2$, $\mathcal{U}_2$ in analogy to the above, and put $i_{02} = i_{01} \cdot i_{12}$. We obtain a sequence of inner models $V_{(n)}$. Let $V_{(\omega)}$ be the direct limit of those with respect to the maps $i_{nm}$; Gaifman proved that $V_{(\omega)}$ is well-founded and (identifying it with its transitization) that the $\kappa_n$'s go under the canonical embeddings to $\kappa_\omega = \mathrm{df} \sup_{n<\omega} \kappa_n$, and the $\mathcal{U}_n$'s to an ultrafilter $\mathcal{U}_\omega$ in $V_{(\omega)}$ on $\kappa_\omega$. Let $N$ be the intersection of the $V_{(n)}$, $n < \omega$. $N \supseteq V_{(n)}$. Bukovský showed that $N$ is a model of ZF + AC, and further that it is a generic extension of $V$ioi. It follows from Mathias [S90] that the sequence $\langle \kappa_n | n < \omega \rangle$ is Příkry generic over $V_{(\omega)}$ with respect to $\mathcal{U}_\omega$. DEHORNOY [S16] has proved that in fact

$$N = V_{(\omega)}[\langle \kappa_n \mid n < \omega \rangle].$$

In fact, iterations $V_{(\omega)}$ can be defined for all ordinals $\zeta$. Call (by T 2016) *$M$ the $\kappa$-model* if $M$ is an inner model of ZF + AC + $V = L^{\mathcal{U}}$ in which $\kappa$ is the mesurable cardinal. KUNEN [S71] showed that if the $\kappa$- and $\lambda$-models exist and $\kappa < \kappa$, then the $\lambda$-model is an iterated ultrapower of the $\kappa$-model. Let $Q$ be the intersection of all $\kappa$-models as $\kappa$ runs through all appropriate ordinals. If $\kappa$-models exist, $Q$ coincides with the *core model* defined and studied by DODD and JENSEN in [S21]. The definition of the core model is via fragments, called *mice*, of which well-founded ultrapowers can be formed and iterated; and "works" whether there are $\kappa$-models or not. The following results are quoted from the manuscript of Dodd and Jensen. If $O^{\#}$ (D 2037) does not exist, the core model is equal to $L$. If there is no $\kappa$-model for any $\kappa$, the covering lemma holds over the core model: that is, given an uncountable set $X$ of ordinals, there is a set $M \supseteq X$ with $X = Y$ and $Y$ in the core model. If $O^{+}$ (D 2040) does not exist but there is a $\kappa$-model, let $\kappa$ be minimal such and $\mathcal{U}$ normal on $\kappa$ in $L^{\mathcal{U}}$: then, still assuming AC, either the covering lemma holds for $L^{\mathcal{U}}$ or there is a Příkry generic sequence $C$ over $L$-- such that the covering lemma holds for $L^C = L^{\mathcal{U},C}$.

Kunen showed that if there is a non-trivial embedding of $L$ into itself which is elementary, or at least preserves $\Sigma_1$ predicates, $O^{\#}$ exists. Dodd and Jensen have also shown that if there is a non-trivial $\Sigma_1$ embedding of the core model into itself, then there is a $\kappa$-model, though no such $\kappa$ need be equal to the first ordinal moved by the embedding.

T 2021 has been strengthened by Solovay to "has $2^{2^\kappa}$ normal measures."

P 2022 is now known to be impossible: MAGIDOR [S80] has shown that if Con[ZFC + SCC) then

Con(ZFC + SCC + the first measurable cardinal is strongly compact).

T 2025 should read "If there is a uniform $\kappa$-complete ultrafilter on $\lambda^+$, then . . ."

P 2028. False: see KUNEN [S72] who shows that the existence of a $\kappa$-complete uniform ultrafilter on $\kappa^+$ implies the conclusion of T 2030.

P 2031 is answered by MAGIDOR [S80]: if Con(ZFC + SuCC) then

Con(ZFC + SuCC + the first strongly compact cardinal is supercompact).

Thus from Magidor's work on P 2022 and P 2031, the first strongly compact may be either equal to the first measurable cardinal or to the first supercompact cardinal, which by T 2020 is much larger then the first measurable. For further indecision in the ordering of large cardinals, see MORGENSTERN [S98] where it is shown that the first huge cardinal may be either larger or smaller than the first strongly compact cardinal.

T 2033. See SILVER [S117].

T 2036. See ROWBOTTOM [S110].

P 2047. Kunen [S73] has shown that if there are $\omega_1$ measurable cardinals, AC is false in Chang's model $M^{\mathrm{On}}_{\omega_1\omega_1}$. See CHANG [$*$, I].

T 2050. See SOLOVAY [$*$, I]. KUNEN has shown in [S71] that if $I$ is a normal $\kappa$-complete $\kappa^+$-saturated ideal on $\kappa$ then $I \cap L^{\mathfrak{B}}$ is prime in $L^{\mathfrak{B}}$. That may fail if $I$ is not normal: see WAGON [S131].

## § 1.

D 2052. The following table is taken from KUNEN [S74].

Let $S^*(\kappa, \lambda)$ mean that there is a $\kappa$-complete ideal on $\kappa$ which is $\kappa$-saturated but not $\lambda'$-saturated for any $\lambda' < \lambda$. Then for $\kappa$ not measurable, the following is known:

| | $\omega < \lambda < \kappa$ | $\lambda = \kappa$ | $\lambda = \kappa^+$ |
|---|---|---|---|
| $\kappa$ a successor cardinal | FALSE ULAM [S129] | FALSE ULAM [S129] | Consistent KUNEN [S74] |
| $\kappa$ weakly but not strongly inaccessible | Consistent PŘÍKRY [S105] | Consistent KUNEN, PARIS [S75] | Consistent |
| $\kappa$ strongly inaccessible but not weakly compact | FALSE TARSKI [S125] | Consistent KUNEN [S74] | Consistent Boos [S10] |
| $\kappa$ strongly inaccessible and weakly compact | FALSE TARSKI [S125] | FALSE LÉVY, SILVER | Consistent KUNEN, PARIS [S75] |

P 2055. A consistency proof is implicit in KUNEN [S74].

P 2057. Done by KUNEN [S74] assuming the consistency of a huge cardinal. Can it be done with less?

T 2059. For a discussion of possible refinements of this result, see BAUMGARTNER, HAJNAL and MÁTÉ [S4].

T 2061. No (WILLIAMS [S132], Chapter II, Theorem 3.4).

Let us note here the theorem of SOLOVAY [S121] that if $\kappa$ and $\lambda$ are cardinals with $\kappa \leq \lambda$ and there is a normal measure on $S_\kappa(\lambda^+)$, then $\square_\lambda$ fails.

For partition theorems in the presence of the axiom of choice, see the forthcoming encyclopaedic book of ERDŐS, HAJNAL, MÁTÉ and RADO [S25]. For partitions in $L$, see SHORE [S115]. For generalizations of $\diamondsuit$ to large cardinals, see JECH [S49]; for results relating weak compactness in various languages, see BELL [S6]; and for more on compactness, see G. V. CHOODNOVSKY [S13].

T 2109. See SILVER [S118] and JENSEN [*, II].

P 2114. MENAS in his thesis [S94] showed that from an appropriate assumption the consistency of the existence of a supercompact cardinal with the GCH and the statement that all sets are rodinal definable may be proved.

T 2215. The proto-argument for Silver's work mentioned in the note to P1203.

P 2117. No (SILVER, assuming Con(ZF + AC + there is an extendible cardinal)).

T 2118. MATHIAS [S90] has proved that a monotone function $f: \omega \to \varkappa$ is Příkry generic over an inner model $M$ with respect to a normal measure $\mu$ on $\varkappa$ in $M$ iff for each $A$ with $\mu(A) = 1$, $\forall n \wedge m > n(f(m) \in A)$.

P 2121. Solovay has proved that if $\varkappa$ is strongly compact and $\lambda$ is a singular strong limit cardinal greater than $\varkappa$ then $2^\lambda = \lambda^+$. See SOLOVAY [S121] which was partly based on the theorem of KETONEN [S67] that if for each $\lambda$ greater then $\varkappa$ there is a $\varkappa$-complete uniform ultrafilter on $\lambda$, then $\varkappa$ is strongly compact.

T 2123 though true, is not due to Kunen but follows from T 2115 and Silver's proof that if

$$(\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \text{there is an extendible cardinal})$$

then

$$\mathrm{Con}(\mathrm{ZF} + \mathrm{AC} + \forall \varkappa(\varkappa 2^{vm} + 2^\varkappa > \varkappa^+))$$

and not by the faulty argument given, which is due to the author's misunderstanding of Kunen's work.

T 2124. See SOLOVAY [*, I]. One direction of this can also be proved by the KUNEN—PARIS method [S75]. Let $\varkappa$ be a measurable cardinal, and $\mathbf{B}$ the measure algebra of $2^\varkappa$. Then $j(\mathbf{B})$ is complete and is the measure algebra of $2^{j(\varkappa)}$, and, (by Fubini's theorem, not by general facts about iterated forcing) the algebra $\mathbf{C}$ in $V^{\mathbf{B}}$ such that $(V^{\mathbf{B}})^{\mathbf{C}} = V^{j(\mathbf{B})}$ is the measure algebra of $2^{j(\varkappa)-\varkappa}$; let $\nu$ be the measure on it. By the main theorem of KUNEN—PARIS [S75] there is in $V^{j(\mathbf{B})}$ a $V^{\mathbf{B}}$-ultrafilter $\mathcal{U}$ on $\varkappa$. Define, in $V^{\mathbf{B}}$, for $A \subseteq \varkappa$,

$$\mu(A) = \nu([\widehat{A} \in U]^{\mathbf{C}}).$$

Then in $V^{\mathbf{B}}$, $\varkappa = 2^\omega$ and $\mu$ is a real-valued measure on $\varkappa$.

In that argument $j$ is any elementary embedding of the universe into an inner model for which $\varkappa$ is the first ordinal moved. As a special case of a general fact about the application of the Kunen—Paris method for obtaining saturated ideals to algebras satisfying the $\varkappa$-chain condition, the filter of sets in $V$ of $\mu$-measure 1 is generated by the normal ultrafilter $\{X \subseteq \varkappa \mid \varkappa \in j(X)\}$ which lies in $V$.

The Kunen—Paris method for forming saturated ideals is in some sense the converse of the Boolean-valued analysis of saturated ideals in KUNEN [S71] and SOLOVAY [*, I].

## Section 3.

The state of knowledge of descriptive set theory has advanced greatly since this *Landscape* was written: see in particular the "Notes on Scales" [S66] by KECHRIS and MOSCHOVAKIS, and MARTIN's forthcoming book.

§ 0.

T 3015. From work of MARTIN and KUNEN it is known that, assuming AD plus DC, each $\aleph_n$ for $n > 2$ is of cofinality $\aleph_2$, $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ are measurable, and each $\aleph_{\omega+n}$ for $n > 2$ is of cofinality $\aleph_{\omega+2}$.

P 3021. Yes (MARTIN [S85]).

T 3022. See MARTIN [S84] HARRINGTON has improved this to: $O^{\#}$ exists iff every $\pi^1_1$ game is determined.

T 3024 is proved in FRIEDMAN [S30]; in fact Friedman had only obtained $\Pi^0_5$ instead of $\Delta^0_5$ but MARTIN has subsequently improved that to $\Pi^0_4$.

T 3025. Now vacuous, as there are no Moschovakis cardinals.

P 3026. Moschovakis has proved $\bigwedge n$ Unif $(\mathbf{\Pi}^1_{2n+1})$ from ZF + AD + AC: more exactly, he has shown that DC + determinacy for all $\mathbf{\Delta}^1_{2n}$ games implies Unif $(\mathbf{\Pi}^1_{2n+1})$. Friedman has proved the following "non-uniform uniformisation theorem": if $V = L$, then every non-empty $\pi^1_2$ set of reals contains a $\Pi^1_2$ singleton. Harrington has proved the converse. See Harrington [S38] for a proof of both.

P 3030. HARRINGTON has shown that *Prewellordering* $(\Sigma^1_3)$ (and hence red $(\Sigma^1)$) holds in Solovay's model of T 3306. See the comment on P 3110.

Let us record here SILVER's interesting result that the set of equivalence classes of a $\mathbf{\Pi}^1_1$ equivalence relation of the reals is either countable or of cardinality $2^{\aleph_0}$.

For more on the axiom of determinancy see FENSTAD [S28], KECHRIS [S65], MARTIN [S86], MARTIN and PARIS [S87], and MOSCHOVAKIS [S100].

§ 1.

For amplification of the remarks following T 3103, see KECHRIS and MOSCHOVAKIS [S66].

T 3105. See JENSEN and SOLOVAY [S59]; for other proofs of this result, see JENSEN [S51] and JENSEN and JOHNSBRÅTEN [S56].

P 3108. Solved affirmatively with $n = 3$ by JENSEN [S51].

P 3110. HARRINGTON [S40] has constructed models of ZF + AC in which the set of constructible reals is, respectively, exactly the following set of reals: $\Delta^1_3$, $\Delta^1_4$, ... $\Delta^1_\omega$ = projective, $\Delta^m_n$ for $1 \leq n \leq \omega$, $2 \leq m \leq \omega$. KECHRIS has observed that in the above model for $\Delta^1_n$, Prewellordering $(\Sigma^1_n)$ and Prewellordering $(\Pi^1_n)$ both fail. By a further argument Harrington can find models in which Sep $(\mathbf{\Sigma}^1_n)$ and Sep $(\mathbf{\Pi}^1_n)$ both fail.

§ 2.

T 3200 is due to HARRISON [S42].

T 3202. See GUASPARI [S36], [S37], KECHRIS [S63] and MANSFIELD [S82]for a discussion of this. Call a set *thin* if it has no perfect subset. Then there is a largest thin $\pi^1_1$ set, which may be defined as $\{x \mid x \in L_{\omega,x}\}$ or as

$$\{x \mid \wedge y(\omega_1^x \leq \omega_1^y \to x \leq_h y)\}$$

or in at least five other ways.

See also the note to T 1240.

T 3205. TRUSS has constructed, without using an inaccessible, a model of set theory in which every set of reals either is of cardinality $\leq \aleph_1$ or has a perfect subset, and in which the constructible reals form a set of cardinality $\aleph_1$.

T 3207. Cf. GUASPARI [S36].

T 3215. The first proof contained a mistake, but MANSFIELD [S79] now has a correct one. For some intermediate stages see References 2 and 4 in [S79]. For a discussion of related questions see HARRINGTON and JECH [S39], [S41].

In their paper [S41], HARRINGTON and JECH prove that it is consistent to have a $\Sigma_1$ well-ordering of the universe, in the sense of LÉVY [B6], without $V$ being equal to $L$. They ask whether the extence of a $\Sigma_1$ well-ordering of $V$ implies the existence of such a well-ordering of length $On$. The answer is yes- let $W_1$ be a $\Sigma_1$ well-ordering of $V$, and $\mathrm{rn}(x)$ the rank of $x$. Obtain a $\Sigma_1$ well: ordering of $V$ of length On by setting

$$x \prec_2 y \leftrightarrow \mathrm{df}\ \mathrm{rn}(x) < \mathrm{rn}(y) \vee (\mathrm{rn}(x) = \mathrm{rn}(y) \wedge x <_1 y).$$

It follows from the unpublished theorem of JENSEN they cite that $V = L$ iff there is a $\Sigma_1$ well-ordering of $V$ of length $On$ such that the function which assigns to each set its set of predecessors in the well-ordering is also $\Sigma_1$.

T 3222. See MANSFIELD [*, I].

§ 3.

T 3302. The proof of this and the next theorem has been illuminated by the application of some results of ROTHBERGER [S109], the principal one being

ZF $\vdash$ If there is a set of reals of power $\varkappa$ which is not first category then the real line is the union of $\varkappa$ sets of measure 0; and similarly with "first category" and "measure 0" interchanged.

T 3306. MATHIAS [S89] has shown that in the model of T 3306, which is described in SOLOVAY [S120] the partition relation $\omega \to (\omega)^\omega$ holds. See also LÉVY and SOLOVAY [S78].

From AD + DC it has been shown (see MARTIN, PARIS [S87]) that $\omega_1 \to (\omega_1)^{\omega_1}$ and $\bigwedge \xi < \omega_2 \colon \omega_2 \to (\omega_2)^{\xi}$ whereas $\omega_2 \to (\omega_2)^{\omega_2}$ is false. Příkry has proved using DC + ADR — the axiom of determinacy for games played with reals — that $\omega \to (\omega)^{\omega}$. See PŘÍKRY [S107], or for related results see MATHIAS [S89].

There is a short proof, due to MARTIN, of $\bigwedge \xi < \omega_1 \colon \omega_1 \to (\omega_1)^{\xi}$ from AD + DC using Jensen's theorem that if $\xi < \omega_1$ and $\langle \xi_\nu \mid \nu < \xi \rangle$ is a sequence of countable ordinals such that each is admissible in the set of its predecessors then for some real $x$ and all $\nu < \xi$, then $\xi_\nu$ is the $(1 + \nu)$-th ordinal admissible in $x$. Jensen's theorem was proved by him using forcing in some unpublished notes on admissible sets. A proof has recently been found by SACKS and HARRINGTON which avoids forcing.

T 3307. See SOLOVAY [S120].

P 3309. An affirmative answer to the following question would assist in solving P 3309, according to recent work of TRUSS:

if a generic or a random real is added to a model of Martin's axiom, is Martin's axiom still true? For generic reals, the answer is No (J. ROITMAN).

T 3315. See the comment to T 3302.

P 3318. Such a model has been found by LAVER [S77].

T 3321. See MARTIN and SOLOVAY [S88].

T 3322. For extensions of this result to higher levels of the projective hierarchy using determinacy for projective games, see KECHRIS [S64].

T 3325. Note that (iii) follows from (i) by the argument of ROTHBERGER [S109]. A propos of RVMs, KUNEN has proved that if $2^{\omega}$ is RVM then it has the $\pi^1_1$ reflection property, whereas in his thesis (Theorem 16.8) he showed that if Con(ZF + AC + MC)

$$\text{Con}(\text{ZF} + \text{AC} + 2^{\omega} \text{ carries an } \omega_1\text{-saturated}$$

$$2^{\omega}\text{-complete ideal but is } \pi^1_1\text{-characterizable}$$

and hence fails to have the $\pi^1_1$ reflection property).

For more on RVMs, see D. V. CHOODNOVSKY [S13].

§ 4.

P 3411. The answer to this and related questions is now known to be intimately connected with the relation of $L$ to $V$. In particular, if $V = L$, the answer to

P 3411 for the case $\varkappa = \omega_1$ (PŘÍKRY [S106]) or $\varkappa = \omega_n$, $n \geq 2$, (JENSEN, KUNEN [S58]) is No. See BENDA, KETONEN [S8], KANAMORI [S60], KETONEN [S68], and JENSEN, KOPPELBERG [S57]. In the last mentioned paper, it is shown as an interesting corollary that if $\neg\, 0^{\#}$, then $\square_\beta$ holds for every singular $\beta$. It was a lemma of KANAMORI in [S60], [S61] that started the wave of results mentioned in the note to P 1203. Did anyone think in 1968 that these two problems were connected?

## Section 4.

### § 0.

D 4000. SOLOVAY has proved that if $V = L$ then $\mathrm{K}(\aleph_\alpha)$ holds for all $\alpha$. See DEVLIN [S17].

T 4004. Solovay's theorem quoted in the last comment generalizes to the result that if $X \subseteq \aleph_{\alpha+1}$ and $V = L[X]$, then $K(\aleph_\alpha)$. It follows that if $K(\aleph_\alpha)$ is false then $\aleph_{\alpha+2}$ is inaccessible in $L$.

T 4006. JENSEN has shown that if $V = L$, or at least if $0^{\#}$ does not exist, then T 4006 holds for the case $\aleph_\beta$ singular as well. Furthermore he has shown that if $V = L$, then for any two $\varkappa$, $\lambda$,

$$\langle \varkappa^{++}, \varkappa\rangle \Rightarrow_0 \langle \lambda^{++}, \lambda\rangle,\ \langle \varkappa^{+++}, \varkappa\rangle \Rightarrow_0 \langle \lambda^{+++}, \lambda\rangle$$

et cetera. See DEVLIN [S17], and for related independence results, see MITCHELL [S96].

D 4007. For more on Rowbottom cardinals and the related notion of Jónsson cardinals see DEVLIN [S18] and KLEINBERG [S69].

KLEINBERG [S70] has proved that if ADD then the $\aleph_n$ are all Jónsson cardinals and $\aleph_\omega$ is a Rowbottom cardinal.

SILVER has proved that if $\aleph_\omega$ is a Jónsson cardinal then it is measurable in an inner model.

T 4013. The hypothesis may be weakened to Con(ZF + AC + there is a Ramsey cardinal). For an account of Silver's proof, see DEVLIN [S19].

KUNEN has shown that if CC holds then $0^{\#}$ exists: see DEVLIN [S17].

### § 1.

See KEISLER, SILVER [*, I].

## Section 5.

§ 0.

For Ackermann's set theory, see LAKE [S76] and REINHARDT [S108].

T 5010. This theorem has been considerable extended by PINCUS [S102], [S103]. See also JECH [S47].

For limitations to the Fraenkel—Mostowski method, see HOWARD [S43].

§ 1.

For a recent review of the state of knowledge concerning the consistency problem for NF, see BOFFA [S9].

T 5100. See JENSEN [S55].

Finally, the reader is encouraged to consult FRIEDMAN's list [S31] for ideas for further areas of research.

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